THE UNIVERSITY OF ALBERTA

DIMENSION THEORY, ISOMORPHISM AND RELATED PROBLEMS IN GROUP RINGS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES & RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1972

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES & RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies & Research for acceptance, a thesis entitled DIMENSION THEORY, ISOMORPHISM AND RELATED PROBLEMS IN GROUP RINGS submitted by MICHAEL M. PARMENTER, B.Sc. (Hons.), in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

In this thesis, we study several problems concerning group rings over arbitrary coefficient rings.

Chapter 1 investigates the dimension subgroups of G over general coefficient rings R. We show that the n'th dimension subgroup of G over R can be expressed in terms of the dimension subgroups over Z and Z_{p^e} for suitable p^e.

In chapter 2, the result on dimension subgroups is applied to the study of residual nilpotence and to the intersection theorem as it applies to the augmentation ideal of a group ring.

In chapter 3, we study the units and Jacobson radical of certain very special group rings. In the case of units, we are concerned with right-ordered groups. In the case of the Jacobson radical, we restrict ourselves even further to G infinite cyclic.

These results are used in chapter 4, where we study the problem of whether or not $R\langle x\rangle \simeq S\langle x\rangle$ implies $R \simeq S$, where $\langle x\rangle$ is an infinite cyclic group.

Chapter 5 deals with problems pertaining to the usual isomorphism question, namely, when does $RG \simeq RH$ imply $G \simeq H$. We first show that for certain integral

domains and finite groups G, the normal subgroups of G can be "located" equationally in RG. We also obtain, by studying the units of Z_pG, isomorphism invariants of Z_pG when G is a finite p-group.

In chapter 6, we present a non-Archimedian analogue of the well known $\ell_1(G)$. When R is a valuation ring, complete with respect to the induced metric, we extend the usual definition of a group ring to allow infinitely many group elements with non-zero coefficients, providing the coefficients approach zero. We investigate the algebraic properties of these extended group rings in the case where R is the field of p-adic numbers or the ring of p-adic integers.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. S.K. Sehgal, for his generous encouragement and assistance, Dr. I.B.S. Passi for his helpful suggestions concerning the material in chapters 1 and 2 and the National Research Council of Canada for their support.

I would also like to thank Miss Linda Hawreliak for typing this thesis.

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INTRODUCTION

We are concerned in this work with several problems concerning group rings over arbitrary coefficient rings. If a ring R has characteristic zero, we know that the group ring RG can be realized as $R \otimes_{Z} ZG$. It is not clear, however, how certain very fundamental structures, such as dimension subgroups for example, are affected by this procedure, and it is this type of question which we try to answer.

In Chapter 1, we study the dimension subgroups of G for general coefficient rings R. The main result of this chapter (Theorem 1.2) gives the structure of the n'th dimension subgroup of G over R in terms of the dimension subgroups over Z and Z_{p^e} for suitable p^e. In this way, we show that the dimension subgroups are dependent only on the primes p satisfying $p^n \in p^{n+1}R$, and the smallest values of such n. This result was previously obtained by Sandling [34] under the assumption that the dimension subgroup conjecture holds.

The primary technique in Chapter 1 is an application of the polynomial maps concept of Passi [27] to the problem at hand. This, together with some calculations involving invertible primes in certain homomorphic images of R, gives the required result. At the end of the chapter, we present a first application of the main result

to the study of free presentations of groups.

In Chapter 2, we apply the results of Chapter 1 to the study of residual nilpotence and to the intersection theorem. More precisely, we examine when $\bigcap_n \Delta_R^n(G) = 0$ and when $[\bigcap_n \Delta_R^n(G)](1 - i) = 0$ for some $i \in \Delta_R(G)$.

The study of residual nilpotence was taken up in earnest by Gruenberg [15]. Our main result (Theorem 2.5) says that if G is finitely generated, then $\bigcap_n \Delta_{\hat{Z}_p}^n(G) = 0 \Leftrightarrow \bigcap_n D_{n,\hat{Z}_p}(G) = 1 \Leftrightarrow G$ is residually a finite p-group, where $\hat{Z}_p$ is the ring of p-adic integers. This implies the result of Bovdi [4], who showed that if G is finitely generated and contains a generalized p-element, then $\bigcap_n \Delta_Z^n(G) = 0$ if, and only if, G is residually a finite p-group. We also extend a theorem of Mital [25] from the integers to commutative rings R satisfying $\bigcap_n p^n R = 0$.

The problem of determining when $\bigcap_n \Delta_R^n(G)$ annihilates $1 - i$ for some $i \in \Delta_R(G)$ has previously been investigated by Smith [42]. Our main result here (Theorem 2.13) gives necessary and sufficient conditions for such an i to exist when $R = Z$ and G is a finitely generated group with a torsion element. Smith proved the result when, in addition to the above restrictions, G is assumed to be nilpotent. We also obtain necessary and

sufficient conditions for such an i to exist when $R = \hat{Z}_p$ and G is finitely generated, and when R is an arbitrary commutative integral domain and G is finite.

In Chapter 3, we study the units and Jacobson radical of certain group rings. In the case of units, we are concerned with RG where G is a right-ordered group. In the case of the Jacobson radical, we are only concerned with G infinite cyclic. These rather specialized problems are studied here for two reasons. Firstly, we wish to see how the techniques and results of Coleman and Enochs [9] and of Amitsur [2] for polynomial rings carry over to group rings. Secondly, some of the results are required in Chapter 4 for the study of isomorphism problems.

As far as units are concerned, our main result (Theorem 3.1) shows that if G is right-ordered, then the units of RG are "trivial" if, and only if, R has no non-zero nilpotent elements. In this case, the situation looks very much like the polynomial ring result. However, the situation is much different when one investigates automorphisms of the infinite cyclic group ring $R\langle x\rangle$, and we prove one result to illustrate this.

Our goal in the case of the Jacobson radical was to obtain an analogue of the result of Amitsur [2] for polynomial rings. We offer some progress on this quest in the second part of Chapter 3. In particular, we find

an equivalent formulation of the conjecture that $J(R\langle x\rangle) = (J(R\langle x\rangle) \cap R)\langle x\rangle$.

Chapter 4 deals with an isomorphism problem for group rings, namely, when does $RG \simeq SG$ imply $R \simeq S$. Again, our particular interest is in the case where $G = \langle x\rangle$, an infinite cyclic group, and in the comparison of this problem with the polynomial ring problem $R[x] \simeq S[x] \overset{?}{\Rightarrow} R \simeq S$. The main result here (Theorem 4.4) says that if R and S have perfect centres, then $R\langle x\rangle \simeq S\langle x\rangle$ implies $R \simeq S$. We also show that if R and S have no nontrivial idempotent or nilpotent elements, then $R\langle x\rangle \simeq S\langle x\rangle$ implies R can be embedded in S and S can be embedded in R. Finally, we show that Hochster's example of two non-isomorphic integral domains with isomorphic polynomial rings does not yield isomorphic infinite cyclic group rings. In particular, $R[x] \simeq S[x]$ does not necessarily imply $R\langle x\rangle \simeq S\langle x\rangle$.

The fundamental technique used in this chapter is the observation that if $\sigma: R\langle x\rangle \to S\langle x\rangle$ is an isomorphism, then *sometimes* an S-algebra automorphism β of $S\langle x\rangle$ can be found such that $\beta\sigma(x) = x$.

In Chapter 5, we consider the more usual isomorphism problem, namely, what does $RG \simeq RH$ have to say about G and H. As in Chapters 1 and 2, part of our interest is in extending the integral group ring results

to more general coefficient rings. Sehgal [38] showed that a finite normal subgroup H of a group G can be identified in the group ring (actually $\hat{H} = \sum_{h \in H} h$ is located) by the conditions x central, $x^2 = mx$, $m \neq 0$, $x_1 = 1$. We show that this result extends to a class of commutative integral domains large enough to contain $Z_p^{\hat{}}$ when G is a p-group.

We also show that in a very special case non-normal Sylow subgroups can be identified in the above manner.

Also in this chapter, we extend a result of Sehgal [37] concerning central units in Z_pG, where G is a finite p-group, from p-groups with property C to arbitrary finite p-groups. As a consequence of this, we show that certain numbers ℓ_i are isomorphism invariants where ℓ_i is the number of conjugacy class sums K of G such that $K^{p^i} \neq 0$ in Z_pG.

Chapter 6 is concerned with a somewhat different problem than the other chapters. We define $\ell(Q_p^{\hat{}},G) = \{\sum \alpha_g g \mid \alpha_g \in Q_p^{\hat{}}, \sum \alpha_g < \infty\}$, i.e. the sum converges with respect to the valuation on $Q_p^{\hat{}}$ where $Q_p^{\hat{}}$ is the field of p-adic numbers, and we study the algebraic properties of this ring and of its subring $\ell(Z_p^{\hat{}},G)$. In other words, the usual idea of a group ring is extended to allow infinite sums dependent on the convergence in the

coefficient ring. The corresponding structure over the complex numbers is the well known $\ell_1(G)$. The usual group ring proofs yield many results about $\ell(\hat{Q}_p, G)$ and $\ell(\hat{Z}_p, G)$, and such arguments are either omitted or only briefly sketched in the chapter. Other problems require different proofs, the most useful technique being the observation that $\ell(\hat{Z}_p, G) \simeq \varprojlim Z_{p^n}G$.

Notation

Throughout this work, R will be a (not necessarily commutative) ring with 1 and G will be a group. The group ring RG is defined to be the set of all finite sums $\sum r_g g$ with r_g in R and $g \in G$ with pointwise equality, addition defined by $\sum r_g g + \sum s_g g = \sum (r_g + s_g) g$ and multiplication defined by $(\sum r_g g)(\sum s_h h) = \sum_k (\sum_{gh=k} r_g s_h) k$.

With any group ring RG, there is associated an augmentation homomorphism $\varepsilon : RG \to R$ defined by $\varepsilon(\sum r_g g) = \sum r_g$. The kernel of this homomorphism is called the augmentation ideal of the group ring and will be denoted by $\Delta_R(G)$. By definition, $\Delta_R(G) = \{\sum r_g g | \sum r_g = 0\}$. Equivalently, $\Delta_R(G)$ could be defined as the ideal generated by $\{g - 1 | g \in G\}$. If H is a subgroup of G, the 2-sided ideal of RG generated by $\{h - 1 | h \in H\}$ will be denoted by $\Delta_R(G,H)$.

We recall here two very fundamental results which will be required in future chapters:

<u>Lemma</u>. $(R_1 \oplus R_2)G \underset{\sigma}{\cong} R_1 G \oplus R_2 G$.

<u>Proof</u>. Define $\sigma(\sum (r_i, s_i) g_i) = (\sum r_i g_i, \sum s_i g_i)$. Then σ is an isomorphism.

Lemma. $R(G_1 \times G_2) \underset{\sigma}{\cong} (RG_1)G_2$.

Proof. Define $\sigma(\sum r_i(g_i,h_i)) = \sum (r_i g_i)h_i$. Then σ is an isomorphism.

CHAPTER 1

Dimension Subgroups Over Arbitrary Coefficient Rings

The n'th dimension subgroup $D_{n,R}(G)$ of the group G with respect to R is defined to be $\{g \in G | g - 1 \in \Delta_R^n(G)\}$. This definition is based on the concept of dimension subgroups of free groups introduced by Magnus [23] and Zassenhaus [44]. The following fact about dimension subgroups is well known:

Lemma 1.1. $[D_{n,R}(G), D_{m,R}(G)] \subseteq D_{n+m,R}(G)$ for all n,m.

Proof. Let $g \in D_{n,R}(G)$ and $h \in D_{m,R}(G)$. Then $[g,h]-1 = ghg^{-1}h^{-1}-1 = (gh - hg)g^{-1}h^{-1} = ((g - 1)(h - 1) - (h - 1)(g - 1))g^{-1}h^{-1} \in \Delta_R^{n+m}(G)$.

In particular, since $D_{1,R}(G) = G$, we see that $\{D_{n,R}(G)\}$ form a descending central series for G, and $G_n \leq D_{n,R}(G)$ where G_n is the n'th term of the lower central series of G.

In the case that $R = Z$, the ring of rational integers, a long-standing question has been whether or not $D_{n,Z}(G) = G_n$ for all n. The statement that this is in fact true is the so-called dimension subgroup conjecture, and the exciting history of this problem can be found in [28].

In this chapter, we determine the structure of dimension subgroups with respect to arbitrary coefficient

rings in terms of subgroups of the type $D_{n,Z}(G)$ and $D_{n,Z_{p^e}}(G)$ where Z_{p^e} is the ring of integers module p^e. Our theorem is the following:

Theorem 1.2. Let R be a ring with 1 and let char $R = 0$. Then $D_{n,R}(G)$ is the product of all $D_{n,Z_{e(p)}}(G) \cap T_p$ (G mod $D_{n,Z}(G)$) over all primes p satisfying $p^n R = p^{n+1} R$ for some n, and where $e(p)$ is the smallest such p^n. If no such prime p exists, then $D_{n,R}(G)$ is equal to $D_{n,Z}(G)$.

Note. If π is any collection of rational primes, then $T_\pi(G \bmod D_{n,Z}(G))$, the set of elements $g \in G$ such that $g^\alpha \in D_{n,Z}(G)$ for some number α divisible only by primes in π, is a subgroup of G (see, for example, Scott [36]).

Theorem 1.2 was proved for commutative rings R by Sandling [34] under the assumption that the dimension subgroup conjecture holds.

The char $R \neq 0$ case has already been completely solved by Sandling [34]:

Theorem 1.3. If R is a ring with 1 and char $R = r \neq 0$, then $D_{n,R}(G) = D_{n,Z_r}(G)$ for all n, where Z_r is the ring of integers modulo r.

Note that Sandling stated the theorem for commutative rings R, but the same proof goes through in the non-commutative case.

Before we present the proof of Theorem 1.2, we will recall several preliminary results.

First recall [27] that a map $f:G \to Q/Z$ is called a Z-polynomial map of degree $\leq n - 1$ if the Z-linear extension of f to ZG vanishes on $\Delta_Z^n(G)$. The following result can be found in [28]:

Lemma 1.4. Let $H = \langle x-1 + \Delta_Z^n(G) \rangle$ be a cyclic subgroup of $\frac{\Delta_Z(G)}{\Delta_Z^n(G)}$ where $x \in G$. Let p be a rational prime. Then if either H is torsion-free or $p \mid |H|$, there exists a Z-polynomial map $f:G \to Q/Z$ such that the order of $f(x)$ equals p in Q/Z.

Proof. Since Q/Z contains elements of all finite orders, we can construct a homomorphism $\alpha : H \to Q/Z$ such that $\alpha(x-1 + \Delta_Z^n(G))$ has order p (by the assumptions on H). Since Q/Z is divisible, we can extend α to

$$\bar{\alpha} : \frac{\Delta_Z(G)}{\Delta_Z^n(G)} \to Q/Z$$

[22]. Define $f : G \to Q/Z$ by $f(g) = \bar{\alpha}(g-1 + \Delta_Z^n(G))$. Then f is a Z-polynomial map of degree $\leq n - 1$ and the order of $f(x) = f(x - 1) = \alpha(x-1 + \Delta_Z^n(G))$ is equal to p. This completes the proof.

Lemma 1.5. Let $R = R_1 \oplus R_2$. Then $g \in D_{n,R}(G)$ if, and only if, $g \in D_{n,R_1}(G)$ and $g \in D_{n,R_2}(G)$.

Proof. By a lemma given before, we know that $(R_1 \oplus R_2)G \simeq R_1G \oplus R_2G$. Hence, by taking homomorphic images, $g \in D_{n,R}(G)$ implies $g \in D_{n,R_1}(G)$ and $g \in D_{n,R_2}(G)$. Conversely, if $g \in D_{n,R_1}(G)$ and $g \in D_{n,R_2}(G)$, then the above isomorphism yields $g \in D_{n,R}(G)$.

The following lemma appears in a more general form in [17]:

Lemma 1.6 (P. Hall). Let Q be the field of rational numbers. Then $D_{n,Q}(G) = T\ (G \bmod D_{n,Z}(G))$, the torsion subgroup of G modulo $D_{n,Z}(G)$.

Proof. Let $x \in T\ (G \bmod D_{n,Z}(G))$. Then $x^k - 1 \in \Delta_Z^n(G)$ for some k. Using the binomial theorem, $(x - 1 + 1)^k - 1 = k(x - 1) + \ldots + (x - 1)^k \in \Delta_Z^n(G)$, so $x - 1 = -\frac{1}{k}\binom{k}{2}(x - 1)^2 + \ldots - \frac{1}{k}(x - 1)^k + s$ where $s \in \Delta_Q^n(G)$. Hence $x - 1 \in \Delta_Q^2(\langle x\rangle) + \Delta_Q^n(G)$, and this implies by iteration that $x - 1 \in \Delta_Q^n(G)$ as required.

Conversely, if $x - 1 \in \Delta_Q^n(G)$, then $z(x - 1) \in \Delta_Z^n(G)$ for some integer z and $x^{z^m} - 1 \in \Delta_Z^n(G)$ for suitable m. This completes the proof.

The proof of the next lemma is the same as the proof of Theorem 1.3. We are not presenting either here because the arguments are indicated in Sandling [34] and are not required again in this work.

Lemma 1.7. If R is a ring with 1 containing Q, then $D_{n,R}(G) = D_{n,Q}(G)$.

We are now ready to tackle the proof of Theorem 1.2.

Proof of Theorem 1.2.

Case I. Assume that $(R,+)$ is torsion-free to begin with.

In this situation, $p^n R = p^{n+1} R$ implies that p is a unit in R and hence $e(p) = 0$ for all p. Hence the result becomes $D_{n,R}(G) = T_\pi \;(G \bmod D_{n,Z}(G))$ where π is the set of all primes invertible in R.

If π consists of all primes, then Lemmas 1.6 and 1.7 say that $D_{n,R}(G) = T\;(G \bmod D_{n,Z}(G))$, the torsion subgroup of G modulo $D_{n,Z}(G)$. Hence we may assume there is a prime not in π.

Let $x \in T_\pi \;(G \bmod D_{n,Z}(G))$. Then $x^q - 1 \in \Delta_Z^n(G)$ for some q divisible only by primes in π. Hence $q(x - 1) + \ldots + (x - 1)^q \in \Delta_Z^n(G) \subseteq \Delta_R^n(G)$ and, since q is invertible in R, $x - 1 \in \Delta_R^n(G)$, hence $x \in D_{n,R}(G)$.

Now assume that $x \in D_{n,R}(G)$, i.e. $x - 1 \in \Delta_R^n(G)$. Let $Z^{-1}R$ be the ring of fractions constructed from R by using the rational integers as denominators (this is acceptable since $(R,+)$ is torsion-free). Form the R-module $Z^{-1}R/R$ and define $\alpha : Q \to Z^{-1}R/R$ by $\alpha(q) = q + R$. This makes sense since we have a copy of Q in $Z^{-1}R$.

Then $\mathrm{Ker}\ \alpha = \{q | q \in R\} = \{\frac{a}{b} | b$ is divisible only by primes in $\pi\}$ where $\frac{a}{b}$ is considered in lowest terms. Hence $Z \subseteq \mathrm{Ker}\ \alpha$ and we have an induced Z-homomorphism $\bar{\alpha} : Q/Z \to Z^{-1}R/R$ with $\mathrm{Ker}\ \bar{\alpha} = \{\frac{a}{b} + Z | b$ is divisible only by primes in $\pi\}$.

Let $f: G \to Q/Z$ be any Z-polynomial map of degree $\leq n - 1$ and denote the Z-linear extension of f to ZG by f also. Let $y = \sum r_i(x_{i_1} - 1) \ldots (x_{i_n} - 1)$ belong to $\Delta_R^n(G)$. Thinking of $\bar{\alpha} \circ f$ extended R-linearly, we see that $(\bar{\alpha} \circ f)(y) = 0$ since $\bar{\alpha}$ is a group homomorphism and f vanishes on $\Delta_Z^n(G)$. Hence $\bar{\alpha} \circ f$ vanishes on $\Delta_R^n(G)$ and, in particular, $f(x - 1)$ belongs to $\mathrm{Ker}\ \bar{\alpha}$ and has order divisible only by primes in π. Using Lemma 1.4, we conclude that there exists a number k_1 divisible only by primes in π such that $k_1(x - 1) \in \Delta_Z^n(G)$. If this were not true, Lemma 1.4 would assert the existence of a polynomial map $f: G \to Q/Z$ such that the order of $f(x)$ is relatively prime to all primes in π (here we use the fact that there exists a prime not in π), and this contradicts $f(x - 1) \in \mathrm{Ker}\ \bar{\alpha}$.

When $n = 1$, the theorem holds trivially since $D_{1,Z}(G) = G$, so assume $n > 1$. In that case, we have $x^{k_1} - 1 = k_1(x - 1) + \ldots + (x - 1)^{k_1} \in \Delta_Z^2(G)$. For $n = 2$, the theorem is then proved. If $n > 2$, repeat the argument with x^{k_1}, which is in $D_{n,R}(G)$, and obtain a number k_2,

divisible only by primes in π, with $k_2(x^{k_1} - 1) \in \Delta_Z^n(G)$.

Therefore, $x^{k_1 k_2} - 1 = k_2(x^{k_1} - 1) + \ldots + (x^{k_1} - 1)^{k_2} \in \Delta_Z^{\min(4,n)}(G)$. Continuing this argument, we obtain a π-number $\ell = k_1 k_2 \ldots k_t$ with $x^{\ell} - 1 \in \Delta_Z^n(G)$. Hence $x \in T_{\pi}$ $(G \bmod D_{n,Z}(G))$ as required.

Case II. Assume that R satisfies the condition that if p is a prime and $p^n R = p^{n+1} R$, then $R = pR$.

The rings considered in case I are in this category, and, as in case I, the theorem reduces to proving $D_{n,R}(G) = T_{\pi}$ $(G \bmod D_{n,Z}(G))$ where π is the set of invertible primes in R.

Clearly, we can conclude as before that T_{π} $(G \bmod D_{n,Z}(G)) \subseteq D_{n,R}(G)$. Let $I = \{r \in R \mid nr = 0$ for some integer $n\}$. Then I forms a proper ideal of R (proper since char $R = 0$) and $(R/I,+)$ is torsion free. If $g \in D_{n,R}(G)$ then $g \in D_{n,R/I}(G)$. By case I, $D_{n,R/I}(G) = T_{\pi^*}$ $(G \bmod D_{n,Z}(G))$ where π^* is the set of all primes invertible in R/I. We claim that $\pi = \pi^*$, and this will complete the proof since T_{π} $(G \bmod D_{n,Z}(G)) \subseteq D_{n,R}(G) \subseteq D_{n,R/I}(G) = T_{\pi^*}$ $(G \bmod D_{n,Z}(G))$.

Let $q \in \pi^*$. Hence there exists $r \in R$ such that $qr \in 1 + I$. Therefore, there exists an integer n such that $nqr = n$. If $(n,q) = 1$, then choose integers a and b such that $an + bq = 1$. We obtain $1 - bq = an = anqr =$

$(1 - bq)qr$, so $q(r - bqr + b) = 1$ and $q \in \pi$.

If $(n,q) \neq 1$, then $q|n$ since q is a prime. If $n = q^i m$ with $(q,m) = 1$, we obtain $q^{i+1}mr = q^i m$. Again, if $am + bq = 1$, we obtain $(1 - bq)q^{i+1}r = (1 - bq)q^i$ and $q^i \in q^{i+1}R$. By our assumption on R, this implies that q is invertible in R and hence $q \in \pi$. This completes the proof of case II.

Case III. R is an arbitrary ring with 1 of characteristic zero.

Let $\sigma(R) = \{p | p^n R = p^{n+1}R$ for some $n\}$. Let $\pi(R) = \{p | R = pR\}$. First we assume that $\sigma(R) - \pi(R)$ is finite. If $\sigma(R) - \pi(R) = \phi$, we are just in case II. Hence we may assume that there exists a $p \in \sigma(R) - \pi(R)$. Say $p^n R = p^{n+1}R$ and n is the smallest such integer. Then $R \simeq R/p^n R \oplus R/J$ where $J = \{s \in R | p^n s = 0\}$. To see this, consider the map $\alpha : R \to R/p^n R \oplus R/J$ defined by $\alpha(r) = (r + p^n R, r + J)$. This is 1 - 1 since $r \in J$ implies $p^n r = 0$ and $r \in p^n R$ implies $r = p^n s = (p^{n+1}\ell)s$ for some $\ell \in R$ since $p^n R = p^{n+1}R$. In fact, $r = p^n s = (p^{2n}\ell^n)s = \ell^n p^n (p^n s) = \ell^n p^n r = 0$ if $r \in J \cap p^n R$.

Also, α is onto since if $r \in R$, we know that $p^n r = p^{2n}s$ for some s in R and, therefore, $r - p^n s \in J$. Hence, for an arbitrary ordered pair $(r_1 + p^n R, r_2 + J)$, r_1 can be assumed to be in J and r_2 in $p^n R$. Hence $\alpha(r_1 + r_2) = (r_1 + p^n R, r_2 + J)$ and α is onto.

By Lemma 1.5, $g \in D_{m,R}(G)$ if, and only if, $g \in D_{m,R/p^nR}(G)$ and $g \in D_{m,R/J}(G)$ for all m.

We now show that $\sigma(R/J) - \pi(R/J) = \sigma(R) - \pi(R) - \{p\}$ and that $\pi(R/J) = \pi(R) \cup \{p\}$. First note that $p \in \pi(R/J)$ since $p^n = p^{n+1}s$ implies $p^n(1 - ps) = 0$ and hence $1 - ps \in J$. If $q \in \sigma(R)$ and $q \neq p$, then $q \in \sigma(R/J)$. Also, $q \in \sigma(R)$ and $q \in \pi(R/J)$ implies that $qr - 1 \in J$ for some r. This would mean $p^nqr = p^n$ and, if $ap^n + bq = 1$, $(1 - bq)qr = 1 - bq$. Hence $q \in \pi(R)$. Therefore we have that $\sigma(R) - \pi(R) - \{p\} \subseteq \sigma(R/J) - \pi(R/J)$ and $\pi(R) \cup \{p\} = \pi(R/J)$.

Finally, if $q \in \sigma(R/J) - \pi(R/J)$, then $q^u \in q^{u+1}r + J$ for some $r \in R$ and some $u \geq 1$. Hence $p^nq^u = p^nq^{u+1}r$ and, if $ap^n + bq^u = 1$, $(1 - bq^u)q^u = (1 - bq^u)q^{u+1}r$, so $q^u \in q^{u+1}R$. Therefore $q \in \sigma(R)$.

Hence $\sigma(R/J) - \pi(R/J) = \sigma(R) - \pi(R) - \{p\}$. Notice also that if $q \in \sigma(R) = \sigma(R/J)$ and $q \neq p$, then $e(q)$ is the same in both R and R/J.

If $|\sigma(R) - \pi(R)| = 1$, we recall from several paragraphs back that $g \in D_{m,R}(G)$ if, and only if, $g \in D_{m,R/p^nR}(G)$ and $g \in D_{m,R/J}(G)$. Since R/J satisfies the conditions of case II (by the above argument), we know that $D_{m,R/J}(G) = T_{\pi(R/J)}$ (G mod $D_{m,Z}(G)$) and the above proof shows that $\pi(R/J) = \pi(R) \cup \{p\} = \sigma(R)$. By Theorem

1.3, $D_{m,R/p^nR}(G) = D_{m,Z_{p^n}}(G)$. Hence $D_{m,R}(G) = D_{m,Z_{p^n}}(G) \cap T_{\pi(R/J)}(G \bmod D_{m,Z}(G))$. For $q \neq p$, $T_q(G \bmod D_{m,Z}(G)) \subseteq D_{m,Z_{p^n}}(G)$, and from this we see that $D_{m,R}(G)$ has the required form. Thus the theorem is proved when $|\sigma(R) - \pi(R)| = 1$. In general, we proceed by induction on $|\sigma(R) - \pi(R)|$ using the identical argument given above.

To complete the proof, we need to consider the case where $\sigma(R) - \pi(R)$ is infinite. As in case II, let I be the ideal of torsion elements of R. Then $D_{n,R}(G) \subseteq D_{n,R/I}(G) = T_{\pi^*}(G \bmod D_{n,Z}(G))$ where π^* is the set of primes invertible in R/I and $\pi^* \subseteq \sigma(R)$ (see proof of case II). Also, the proof given in the first part of case III shows that $D_{n,R}(G) \subseteq D_{n,Z_{e(p)}}(G)$ for all $p \in \sigma(R)$. Hence we obtain that $D_{n,R}(G)$ is contained in the product of all $D_{n,Z_{e(p)}}(G) \cap T_p(G \bmod D_{n,Z}(G))$, and we need only prove containment in the other direction.

Let x be in the product of all $D_{n,Z_{e(p)}}(G) \cap T_p(G \bmod D_{n,Z}(G))$. Then x is a product of finitely many elements from different terms $D_{n,Z_{e(p)}}(G) \cap T_p(G \bmod D_{n,Z}(G))$ and let $\theta(R)$ be the corresponding *finite* set of primes.

Let $L = \{r \in R \mid zr = 0$ for some integer z divisible only by primes in $\sigma(R) - \theta(R)\}$. L is an ideal of R. We claim that $\sigma(R/L) = \sigma(R)$ and $\pi(R/L) = \pi(R) \cup (\sigma(R) - \theta(R))$. Clearly $\sigma(R) \subseteq \sigma(R/L)$. If $p^n \in p^{n+1}R + L$, then

$zp^n \in zp^{n+1}R$ for some z divisible only by primes in $\sigma(R) - \theta(R)$. If $(z,p) = 1$, we obtain $p^n \in p^{n+1}R$ as before. If $(z,p) \neq 1$, then $p|z$ and $p \in \sigma(R) - \theta(R)$ by hypothesis. Hence $\sigma(R/L) = \sigma(R)$.

If $q \in \pi(R)$, then $q \in \pi(R/L)$, and if $q \in \sigma(R) - \theta(R)$, then $q^n = q^{n+1}r$ implies $(1 - qr)q^n = 0$, so $q \in \pi(R/L)$. Hence $\pi(R) \cup (\sigma(R) - \theta(R)) \subseteq \pi(R/L)$. Conversely, if $s \in \pi(R/L)$, then $sr - 1 \in L$ and $zsr = z$ for some z divisible only by primes in $\sigma(R) - \theta(R)$. Since we may assume $s \in \theta(R)$, we know $(s,z) = 1$ and we get $s \in \pi(R)$ as before. Therefore $\pi(R) \cup (\sigma(R) - \theta(R)) = \pi(R/L)$. Again, notice that if $p \in \theta(R)$, $e(p)$ is the same for both R and R/L.

By the first part of case III, $x \in D_{n,R/L}(G)$ since $\sigma(R/L) - \pi(R/L)$ is finite. Therefore, $x - 1 \in \Delta_R^n(G) + L(G)$ and there exists an integer z divisible only by primes in $\sigma(R) - \theta(R)$ such that $z(x - 1) \in \Delta_R^n(G)$. However, by the definition of $\theta(R)$, there exists some integer m divisible only by primes in $\theta(R)$ such that $m(x - 1) \in \Delta_Z^n(G) \subseteq \Delta_R^n(G)$ since char $R = 0$. Since $(z,m) = 1$, we obtain $x - 1 \in \Delta_R^n(G)$ and $x \in D_{n,R}(G)$ as required. This completes the proof of the result.

<u>Example</u>. The following example will be used several times in the next chapter, so we present it here. If $R = \hat{Z_p}$, the ring of p-adic integers, then $D_{n,\hat{Z_p}}(G) = T_\pi$ (G mod

$D_{n,Z}(G))$ where $\pi = \{q | q \text{ prime}, q \neq p\}$, since every prime but p is invertible in $Z_p^{\hat{}}$ and $p^n \notin p^{n+1}Z_p^{\hat{}}$.

In chapter 2, we will see several applications of Theorem 1.2 to problems involving the augmentation ideal of the group ring. We would like to close this chapter by presenting an application of Theorem 1.2 to the study of free presentations of groups.

If $1 \to K \to F \to G \to 1$ is a non-cyclic free presentation of a finite group G, Gruenberg [15] has shown that $F/[K,K]$ is residually nilpotent if, and only if, G is a p-group by showing that both statements are equivalent to $\bigcap_n \Delta_Z^n(G) = 0$. Mital [25] extended these results to infinite groups with generalized p-elements. Concerning the relationship between $F/[K,K]$ and G, we prove the following which was proved by Passi [26] for the case $R = Z$.

<u>Proposition 1.8</u>. Let $(R,+)$ be torsion free. Then $\bigcap_n D_{n,R}(F/[K,K]) = 1$ implies $\bigcap_n D_{n,R}(G) = 1$ where $1 \to K \to F \xrightarrow{\alpha} G \to 1$ is a free presentation of G.

<u>Proof</u>. Let $g \in \bigcap_n D_{n,R}(G)$. By Theorem 1.2, there exist numbers π_n with $g^{\pi_n} \in D_{n,Z}(G)$ where each π_n is divisible only by primes in π and $\pi = \{p | R = pR\}$.

Therefore, $g^{\pi_n} - 1 \in \Delta_Z^n(G)$ and it follows that there exist π_n' divisible only by primes in π with $\pi_n'(g - 1) \in \Delta_Z^n(G)$.

Thinking of $\bar{K} = K/[K,K]$ as a G-module under the action $\bar{k} \cdot g = \overline{f^{-1}kf}$ where $\alpha(f) = g$, we observe that $\bar{K} \cdot \pi_n'(g - 1) \in \underbrace{[\bar{K} \ldots [\bar{K},\bar{F}] \ldots \bar{F}]}_{n \text{ times}}$ where $\bar{F} = F/[K,K]$.

Hence if $r \in K$ and $\alpha(f) = g$, $\overline{f^{-1}rfr^{-1}}^{\,\pi_n'} \in (\bar{F})_n \subseteq D_{n,Z}(\bar{F})$. Since $\bigcap_n D_{n,R}(\bar{F}) = 1$, we use Theorem 1.2 again to conclude that $\overline{f^{-1}rfr^{-1}} = 1$ and $K/[K,K] \cdot (g - 1) = 1$. Hence we obtain $g = 1$, using the following Theorem due to Auslander and Lyndon [3]:

<u>Theorem 1.9</u>. If $1 \to K \to F \xrightarrow{\alpha} G \to 1$ is a free presentation of an arbitrary group G with $K \neq 1$, then the operation of G on the right G-module $\bar{K}$ is faithful.

CHAPTER 2

Residual Nilpotence and the Intersection Theorem

In this chapter, we apply Theorem 1.2 of chapter 1 to the study of residual nilpotence of the augmentation ideal of a group ring and to the investigation of the intersection theorem as it applies to group rings. In the case of residual nilpotence, we generalize results of Bovdi [4] and Mital [25]. As far as the intersection theorem is concerned, we extend some of the results of Smith [42]. Our main results of this chapter are Theorems 2.5 and 2.13.

1. Residual Nilpotence

Definition. An ideal I of a ring R is called residually nilpotent if, and only if, $\bigcap_n I^n = 0$.

The problem of determining when $\Delta_R(G)$ is residually nilpotent and, more generally, when $\Delta_R^\alpha(G) = 0$ for some ordinal number α, seems to be very difficult. When G is a finitely generated torsion-free nilpotent group, Jennings [21] showed that $\bigcap_n \Delta_Q^n(G) = 0$. Formanek [12] showed that $\bigcap_n \Delta_R^n(G) = 0$ for any ring R when G is finitely generated torsion-free nilpotent.

Gruenberg [15] showed that when G is finite, $\bigcap_n \Delta_Z^n(G) = 0 \Leftrightarrow G$ is a p-group. Bovdi [4] extended this by

showing that if G is finitely generated and contains a generalized p-element, then $\bigcap_n \Delta_Z^n(G) = 0 \Leftrightarrow G$ is residually a finite p-group. Mital [25] generalized this further by showing that if G contains a generalized p-element, then $\bigcap_n \Delta_Z^n(G) = 0 \Leftrightarrow G$ is residually a nilpotent p-group of finite exponent.

<u>Definition</u>. $1 \neq g \in G$ is called a generalized p-element if, for every positive integer n, there exists $r(n)$ such that $g^{p^{r(n)}} \in D_{n,Z}(G)$.

<u>Definition</u>. If $\mathcal{K}$ is a class of groups, then a group G is said to be residually in $\mathcal{K}$, written $G \in R\mathcal{K}$, if there exist normal subgroups N_α of G such that $\bigcap_\alpha N_\alpha = 1$ and $G/N_\alpha \in \mathcal{K}$ for all α.

As an example, Mital's result can be written as $\bigcap_n \Delta_Z^n(G) = 0 \Leftrightarrow G \in R\overline{\mathcal{N}_p}$ where $\overline{\mathcal{N}_p}$ denotes the class of nilpotent p-groups of finite exponent. We shall let $\mathcal{F}_p$ denote the class of finite p-groups.

In this section, we extend the results of Mital and Bovdi to more general coefficient rings. Our main result shows that if G is finitely generated and if R is of characteristic zero and satisfies (i) every prime but p is invertible in R, and (ii) $\bigcap_n p^n R = 0$, then $\bigcap_n \Delta_R^n(G) = 0 \Leftrightarrow \bigcap_n D_{n,R}(G) = 1 \Leftrightarrow G \in R\mathcal{F}_p$.

For example, $R = \hat{Z}_p$ satisfies conditions (i) and (ii).

This shows that we can remove the hypothesis that G has a generalized p-element if we pass to rings of coefficients of a certain type. We also observe the equivalence $\bigcap_n \Delta_R^n(G) = 0 \Leftrightarrow \bigcap_n D_{n,R}(G) = 1$, which is generally not true.

Basic to our study of residual nilpotence is the following theorem of Hartley [18]:

<u>Theorem 2.1</u>. If p is prime and $G \in R\overline{N_p}$ and $\bigcap_n p^n R = 0$, then $\bigcap_n \Delta_R^n(G) = 0$.

<u>Note</u>. Hartley actually only stated this for $G \in \overline{N_p}$, but the argument in [25] shows that the result holds for $G \in R\overline{N_p}$.

Before we prepare for the main theorem, let us extend Mital's result to arbitrary coefficient rings. Our proof differs only very slightly from that given by Mital [25], so will only be sketched.

<u>Proposition 2.2</u>. Let R be commutative with 1 and let G have a generalized p-element. Then $\bigcap_n \Delta_R^n(G) = 0$ if, and only if, $G \in R\overline{N_p}$ and $\bigcap_n p^n R = 0$.

Proof. The implication ($\Leftarrow$) is just Theorem 2.1 and does not require the existence of a generalized p-element.

Conversely, assume $\bigcap_n \Delta_R^n(G) = 0$. Let $D_{n,m,p,R}(G) = \{x \in G \mid x - 1 \in \Delta_R^n(G) + p^m\Delta_R(G)\}$. If $x \in \bigcap_{n,m} D_{n,m,p,R}(G)$ and y is a generalized p-element, then $(x - 1)(y - 1) \in \Delta_R^n(G)$ for all n. This implies that $x = 1$, except for the case when $x = y$, $x^2 = 1$ and char $R = 2$. In the latter situation, however, $D_{n,m,p,R}(G) = D_{n,R}(G)$ whenever $m \geq 1$ since $2\Delta_R(G) = 0$. Hence $\bigcap_{n,m} D_{n,m,p,R}(G) = \bigcap_n D_{n,R}(G) = 1$.

In any case, we conclude that $\bigcap_{n,m} D_{n,m,p,R}(G) = 1$. However, $G/D_{n,m,p,R}(G)$ is a nilpotent p-group of finite exponent. Hence $G \in R\overline{N_p}$.

If $\bigcap_n p^nR \neq 0$, then let $s \in \bigcap_n p^nR$ and note that $s(y - 1) \in \Delta_R^n(G)$ for all n. This is a contradiction.

Smith [42] proved Proposition 2.2 in the case where R is a commutative noetherian domain and G is a finite group.

The requirement that G have a generalized p-element was introduced by Bovdi and it cannot be dispensed with if one considers integral group rings. For example, if $G = (Q,+)$, the additive group of the field of rational numbers, then Hartley [18] has shown that $\bigcap_n \Delta_Z^n(G) = 0$, while G is clearly not in $R\overline{N_p}$ since it is divisible.

We now establish machinery to prove the extension of Bovdi's result.

<u>Definition</u>. Let $\mathcal{K}_{p'}$ be the class of all groups G such that $D_{n,Z}(G) = 1$ for some n and such that G has no elements of finite order relatively prime to p.

<u>Remark</u>. A finitely generated group in $\mathcal{K}_{p'}$ is in $\mathcal{RF}_p$ (see Lemma 2.4).

<u>Proposition 2.3</u>. Let R be of characteristic zero and satisfy (i) every prime but p is invertible in R and (ii) $\bigcap_n p^n R = 0$. Then $G \in \mathcal{RK}_{p'}$ if, and only if, $\bigcap_n D_{n,R}(G) = 1$.

<u>Proof</u>. First assume that $\bigcap_n D_{n,R}(G) = 1$. Now $D_{n,Z}(G/D_{n,R}(G)) = 1$ since $\bar{g} - 1 \in \Delta_Z^n(G/D_{n,R}(G))$ implies $g - 1 \in \Delta_Z^n(G) + \Delta_Z(G,D_{n,R}(G)) \subseteq \Delta_R^n(G)$ and $g \in D_{n,R}(G)$. By Theorem 1.2, $G/D_{n,R}(G)$ has only p-torsion. Hence $G/D_{n,R}(G) \in \mathcal{K}_{p'}$ and $G \in \mathcal{RK}_{p'}$.

Conversely, assume $G \in \mathcal{RK}_{p'}$ but $\bigcap_n D_{n,R}(G) \neq 1$. Choose $x \in \bigcap_n D_{n,R}(G)$. Then there exists $N \triangleleft G$ such that $x \notin N$ and $G/N \in \mathcal{K}_{p'}$. $\bar{x} - 1 \in \bigcap_n D_{n,R}(G/N)$. Hence we may assume $G \in \mathcal{K}_{p'}$.

Again, let $x \in \bigcap_n D_{n,R}(G)$. By Theorem 1.2, $x \in T_\pi$ $(G \bmod D_{n,Z}(G))$ where $\pi = \{q | q \neq p\}$ for all n. Since $G \in \mathcal{K}_{p'}$, $D_{n,Z}(G) = 1$ for some n and, therefore, $x^s = 1$ for some s divisible only by primes in π. But

$G \in \mathcal{K}_{p'}$, implies G has only p-torsion. Hence $x = 1$ and the result holds.

The next is a result of Gruenberg:

Lemma 2.4 [34]. Let G be finitely generated, nilpotent and π-torsion-free where π is a collection of primes and not every prime is in π. Let π' be the set of primes not in π. Then G is residually a finite π'-group, that is, there exist $N_\alpha \triangleleft G$ with $\bigcap_\alpha N_\alpha = 1$ such that G/N_α is finite and every element in G/N_α is of order a π'-number (a number divisible only by primes in π').

We now prove the main result:

Theorem 2.5. Let R be a ring with 1 of characteristic zero satisfying (i) every prime except one, say p, is invertible in R and (ii) $\bigcap_n p^n R = 0$. Let G be finitely generated. Then the following are equivalent:

(i) $\bigcap_n \Delta_R^n(G) = 0$

(ii) $\bigcap_n D_{n,R}(G) = 1$

(iii) $G \in \mathcal{RF}_p$.

Proof. (i) $\Rightarrow$ (ii) is always true. (ii) $\Rightarrow$ (iii) follows from Proposition 2.3 and Lemma 2.4. (iii) $\Rightarrow$ (i) is a special case of Theorem 2.1.

As a model for R we have in mind $\hat{Z}_p$, the ring of p-adic integers.

Corollary 2.6. Let G be finitely generated. Then the following are equivalent:

(i) $\bigcap_n \Delta^n_{\hat{Z}_p}(G) = 0$

(ii) $\bigcap_n D_{n,\hat{Z}_p}(G) = 1$

(iii) $G \in \mathcal{RF}_p$.

Corollary 2.7 [4]. Let G be finitely generated and have a generalized p-element. Then $\bigcap_n \Delta^n_Z(G) = 0$ implies $G \in \mathcal{RF}_p$.

Proof. Let g be a generalized p-element and let $h \in \bigcap_n D_{n,\hat{Z}_p}(G)$. Using Theorem 1.2, we obtain $(g - 1)(h - 1) \in \bigcap_n \Delta^n_Z(G)$. Hence $h = 1$ and $\bigcap_n D_{n,\hat{Z}_p}(G) = 1$. The result follows from Theorem 2.5.

Corollary 2.8. If G is finitely generated, then $\bigcap_n \Delta^n_{Z_p}(G) = 0 \Leftrightarrow \bigcap_n \Delta^n_{\hat{Z}_p}(G) = 0$.

Proof. It is a result of Mal'cev [24] that $\bigcap_n \Delta^n_{Z_p}(G) = 0 \Leftrightarrow G \in \overline{\mathcal{RN}_p}$. Hence, for G finitely generated, $G \in \mathcal{RF}_p$ and the result follows.

Corollary 2.9. If G has a generalized p-element, then $\bigcap_n \Delta^n_Z(G) = 0 \Leftrightarrow \bigcap_n \Delta^n_{\hat{Z}_p}(G) = 0$.

Proof. Let $\bigcap_n \Delta_Z^n(G) = 0$. By Mital's result, $G \in \overline{RN_p}$, and, by Theorem 2.1, this implies that $\bigcap_n \Delta_{\hat{Z}_p}^n(G) = 0$.

2. The Intersection Theorem

Here we are searching for necessary and sufficient conditions for the existence of $0 \neq i \in \Delta_R(G)$ such that $[\bigcap_n \Delta_R^n(G)](1 - i) = 0$. Results in this direction have previously been obtained by Smith [42], who was concerned with generalizing the usual Intersection Theorem for commutative noetherian rings, and who then applied his observations to the case where the ring is a group ring and the ideal its augmentation ideal.

Our main result (Theorem 2.13) gives necessary and sufficient conditions for such an i to exist when $R = Z$ and G is a finitely generated group with a torsion element. Later propositions give necessary and sufficient conditions for the existence of i when G is finitely generated and $R = \hat{Z}_p$ and when G is finite and R is a commutative integral domain.

First we consider the case $R = Z$. We need three preliminary lemmas.

Lemma 2.10. Let $G_1 \times \ldots \times G_n \triangleleft H$ and assume $G_i \triangleleft H$ for $1 \leq i \leq n$. Then $\Delta_Z(H, G_1 \times \ldots \times G_{n-1}) \cap \Delta_Z(H, G_1 \times \ldots \times G_{n-2} \times G_n) \cap \ldots \cap \Delta_Z(H, G_2 \times \ldots \times G_n) =$

$$= \sum_{i \neq j} \Delta_Z(H,G_i)\Delta_Z(H,G_j).$$

Proof. The proof proceeds by induction on n. First take $n = 2$. Certainly $\Delta_Z(G_1)\Delta_Z(G_2) \subseteq \Delta_Z(H,G_1) \cap \Delta_Z(H,G_2)$. Conversely, let $x \in \Delta_Z(H,G_1) \cap \Delta_Z(H,G_2)$. We can write $x = \sum \alpha_{i,j} h_i g_{1i} g_{2i}(g_{1j} - 1)$ where $\alpha_{i,j} \in Z$, the $\{h_i\}$ come from a transversal of $G_1 \times G_2$ in H, $g_{1i} \in G_1$, $g_{2i} \in G_2$ and $g_{1j} \in G_1$ since $x \in \Delta_Z(H,G_1)$.

Hence $x = \sum \alpha_{i,j} h_i g_{1i}(g_{2i} - 1)(g_{1j} - 1) + \sum \alpha_{i,j} h_i g_{1i}(g_{1j} - 1)$. Since $x \in \Delta_Z(H,G_2)$, we conclude that $\sum \alpha_{i,j} h_i g_{1i}(g_{1j} - 1) \in \Delta_Z(H,G_2)$. We can easily conclude that $\sum \alpha_{i,j} h_i g_{1i}(g_{1j} - 1) = 0$ and, therefore, $x \in \Delta_Z(H,G_2)\Delta_Z(H,G_1) = \Delta_Z(H,G_1)\Delta_Z(H,G_2)$ as required.

Now assume that the theorem holds for $n - 1$. Hence $\Delta_Z(H,G_2 \times \ldots \times G_{n-1}) \cap \Delta_Z(H,G_2 \times \ldots \times G_{n-2} \times G_n) \cap \ldots \cap \Delta_Z(H,G_3 \times \ldots \times G_n) = \sum_{\substack{j \neq i \\ i,j \geq 2}} \Delta_Z(H,G_i)\Delta_Z(H,G_j)$ by induction.

By going from H to $\overline{H} = H/G_1$, we conclude from the above remark that $\Delta_Z(H,G_1 \times \ldots \times G_{n-1}) \cap \Delta_Z(H,G_1 \times \ldots \times G_{n-2} \times G_n) \cap \ldots \cap \Delta_Z(H,G_1 \times G_3 \times \ldots \times G_n) \subseteq \Delta_Z(H,G_1) + \sum_{\substack{j \neq i \\ i,j \geq 2}} \Delta_Z(H,G_i)\Delta_Z(H,G_j)$.

Now $\Delta_Z(H,G_1) \cap \Delta_Z(H,G_2 \times \ldots \times G_n) = \Delta_Z(H,G_1)\Delta_Z(H,G_2 \times \ldots \times G_n)$ by induction. Also $\sum_{\substack{j \neq i \\ i,j \geq 2}} \Delta_Z(H,G_i)\Delta_Z(H,G_j) \subseteq \Delta_Z(H,G_2 \times \ldots \times G_n)$. Hence

$\Delta_Z(H,G_1 \times \ldots \times G_{n-1}) \cap \Delta_Z(H,G_1 \times \ldots \times G_{n-2} \times G_n) \cap \ldots \cap \Delta_Z(H,G_2 \times \ldots \times G_n) \subseteq (\Delta_Z(H,G_1) + \sum_{\substack{j\neq i \\ i,j\geq 2}} \Delta_Z(H,G_i)\Delta_Z(H,G_j)) \cap \Delta_Z(H,G_2 \times \ldots \times G_n) = (\Delta_Z(H,G_1) \cap \Delta_Z(H,G_2 \times \ldots \times G_n)) + \sum_{\substack{j\neq i \\ i,j\geq 2}} \Delta_Z(H,G_i)\Delta_Z(H,G_j) = \Delta_Z(H,G_1)\Delta_Z(H,G_2 \times \ldots \times G_n) + \sum_{\substack{j\neq i \\ i,j\geq 2}} \Delta_Z(H,G_i)\Delta_Z(H,G_j) \subseteq \sum_{j\neq i} \Delta_Z(H,G_i)\Delta_Z(H,G_j)$ by observing that $\Delta_Z(H,G_2 \times \ldots \times G_n) \subseteq \sum_{i\geq 2} \Delta_Z(H,G_i)$ using the identity $gh - 1 = g - 1 + h - 1 + (g - 1)(h - 1)$.

Hence $\Delta_Z(H,G_1 \times \ldots \times G_{n-1}) \cap \ldots \cap \Delta_Z(H,G_2 \times \ldots \times G_n) \subseteq \sum_{i\neq j} \Delta_Z(H,G_i)\Delta_Z(H,G_j)$. The containment in the other direction is trivial.

The next lemma is a result of Smith [42] although his proof is somewhat different.

Lemma 2.11. Let G be finite nilpotent. Then there exists $i \in \Delta_Z(G)$ such that $[\bigcap_n \Delta_Z^n(G)](1 - i) = 0$.

Proof. If G is a p-group, Theorem 2.1 says that $\bigcap_n \Delta_Z^n(G) = 0$. Hence we may assume that there exist at least two different primes, p and q, dividing $|G|$.

First assume G is abelian and $o(g) = p^r$, $o(h) = q^s$. Then $ap^r + bq^s = 1$ implies that $(g - 1)(h - 1) = (ap^r + bq^s)(g - 1)(h - 1) = ap^r(g - 1)(h - 1) + (g - 1)bq^s(h - 1) = -a\binom{p^r}{2}(g - 1)^2(h - 1) \ldots$

$-a(g - 1)^{p^r}(h - 1) - b(g - 1)\binom{q^s}{2}(h - 1)^2 \ldots$
$-b(g - 1)(h - 1)^{q^s}$.

Hence $(g - 1)(h - 1)(1 + a\binom{p^r}{2}(g - 1) + \ldots + a(g - 1)^{p^r-1} + b\binom{q^s}{2}(h - 1) + \ldots + b(h - 1)^{q^s-1}) = 0$ since G is nilpotent, and we have found an element $i_{g,h}$ in $\Delta_Z(G)$ with $(g - 1)(h - 1)(1 - i_{g,h}) = 0$. Take $1 - i = \pi(1 - i_{g,h})$ over all such pairs of elements g,h (and all pairs of primes (p,q) dividing $|G|$).

Since G is nilpotent, $G = S_{p_1} \times \ldots \times S_{p_n}$ where S_{p_k} is the p_k-Sylow subgroup of G. For fixed j, $G/ \underset{k\neq j}{\pi} S_{p_k}$ is a p-group and Theorem 2.1 says that this implies $\bigcap_n \Delta_Z^n(G) \subseteq \Delta_Z(G, \underset{k\neq j}{\pi}(S_{p_k}))$ for all j. By Lemma 2.10, we obtain $\bigcap_n \Delta_Z^n(G) \subseteq \sum_{k\neq j} \Delta_Z(G,S_{p_k})\Delta_Z(G,S_{p_j})$. The $1 - i$ chosen in the last paragraph annihilates $\Delta_Z(S_{p_k})\Delta_Z(S_{p_j})$ for all k,j since G is abelian and we obtain $[\bigcap_n \Delta_Z^n(G)](1 - i) = 0$.

We now consider the case where G is not abelian and proceed by induction on $|G|$. Choose z_1,z_2 central in G so that $o(z_1) = p, o(z_2) = q$ with $(p,q) = 1$. By induction, there exist i_1 and i_2 in $\Delta_Z(G)$ such that $[\bigcap_n \Delta_Z^n(G/\langle z_1\rangle)](1 - \overline{i_1}) = 0$ and $[\bigcap_n \Delta_Z^n(G/\langle z_2\rangle)](1 - \overline{i_2}) = 0$ and $\overline{i_1} \neq 0$, $\overline{i_2} \neq 0$.

Hence $[\bigcap_n \Delta_Z^n(G)](1 - i_1)(1 - i_2) \subseteq [\bigcap_n \Delta_Z^n(G)] \cap \Delta_Z(G,\langle z_1\rangle) \cap \Delta_Z(G,\langle z_2\rangle) \subseteq [\bigcap_n \Delta_Z^n(G)] \cap \Delta_Z(G,\langle z_1\rangle)\Delta_Z(G,\langle z_2\rangle)$ by Lemma 2.10.

Using the argument for abelian groups, there exists $j \in \Delta_Z(\langle z_1\rangle \times \langle z_2\rangle)$ with $(z_1 - 1)(z_2 - 1)(1 - j) = 0$. Hence $[\bigcap_n \Delta_Z^n(G)](1 - i_1)(1 - i_2)(1 - j) = 0$ and the result holds with $1 - i = (1 - i_1)(1 - i_2)(1 - j)$.

Lemma 2.12. Let $x \in \Delta_Z(G)$ satisfy $x \in \Delta_{\hat{Z}_p}^n(G)$ for all primes p. Then $x \in \Delta_Z^n(G)$.

Proof. Let π_p be the natural map from Q/Z to $\hat{Q}_p/\hat{Z}_p$ where $\hat{Q}_p$ is the field of p-adic numbers. Let $f:G \to Q/Z$ be any polynomial map of degree $\leq n - 1$. Then $\pi_p \circ f(x) = 0$ for all p (see the argument in Case I of Theorem 1.2) since $x \in \Delta_{\hat{Z}_p}^n(G)$ for all p. However, $\operatorname{Ker} \pi_p = \{\frac{a}{b} + Z \mid b$ is not divisible by $p\}$ and $f(x) \in \operatorname{Ker} \pi_p$ for all p, so $f(x) = 0$ in Q/Z. By Lemma 1.4, $x \in \Delta_Z^n(G)$ as required.

We are now ready to prove the main result for the integral case.

Theorem 2.13. Let G be a finitely generated group with a torsion element and let $\beta = \{p \mid p$ prime, there exists $g \in G$ of order $p\}$. Then the following are equivalent:

(i) There exists $i \in \Delta_Z(G)$ such that

$[\bigcap_n \Delta_Z^n(G)](1 - i) = 0.$

(ii) <u>either</u> $\bigcap_n \Delta_Z^n(G) = 0$ <u>or</u> for all $p \in \beta$,

$\bigcap_n D_{n,\hat{Z}_p}(G)$ is finite and p-torsion-free.

<u>Note</u>. Partial results of this nature are found in Smith [42]. For example, if G is also nilpotent, the result is there.

<u>Proof</u>. First assume (i) holds. If $t \in \bigcap_n D_{n,Z}(G)$, then $(t - 1)(1 - i) = 0$, so $\bigcap_n D_{n,Z}(G)$ is finite and $1 - i \in (\sum_{g \in \bigcap_n D_{n,Z}(G)} g)ZG$. This implies that the content 1 of $1 - i$ is divisible by $|\bigcap_n D_{n,Z}(G)|$ in Z, so $\bigcap_n D_{n,Z}(G) = 1$.

Let $g \in G$ be of order p and let $h \in \bigcap_n D_{n,\hat{Z}_p}(G)$. Then, because the order of h modulo $D_{n,Z}(G)$ is relatively prime to p (Theorem 1.2), $(h - 1)(g - 1) \in \bigcap_n \Delta_Z^n(G)$ and hence $(h - 1)(g - 1)(1 - i) = 0$. As in the above paragraph, $(g - 1)(1 - i) \neq 0$ so we conclude that $\bigcap_n D_{n,\hat{Z}_p}(G)$ is finite and, since $\bigcap_n D_{n,Z}(G) = 1$, Theorem 1.2 says that $\bigcap_n D_{n,\hat{Z}_p}(G)$ is p-torsion-free. This argument works for any p such that there exists $g \in G$ of order p.

Now assume that (ii) holds and $\bigcap_n \Delta_Z^n(G) \neq 0$. By Theorem 1.2, $G/D_{n,\hat{Z}_p}(G)$ is finitely generated, nilpotent and contains no elements of order relatively prime to p. Hence, by Lemma 2.4, $G/D_{n,\hat{Z}_p}(G) \in \mathcal{RF}_p$ and, by Theorem 2.1,

$\bigcap_m \Delta_Z^m(G/D_{n,\hat{Z}_p}(G)) = 0$. Hence $\bigcap_m \Delta_Z^m(G) \subseteq \Delta_Z(G, D_{n,\hat{Z}_p}(G))$ for all n.

We claim that this implies $\bigcap_m \Delta_Z^m(G) \subseteq \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G))$. Say to the contrary that $x \in \bigcap_m \Delta_Z^m(G) - \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G))$. Then $\bar{x} \neq 0$ in $Z(G/\bigcap_n D_{n,\hat{Z}_p}(G))$ and, if $\bar{x} = \sum \alpha_{g_i} g_i$ with $g_i \in G/\bigcap_n D_{n,\hat{Z}_p}(G)$, then choose m so that $g_i g_j^{-1}$ and g_i do not belong to $\overline{D_{m,\hat{Z}_p}(G)}$ for all $i,j (i \neq j)$. We obtain $\bar{\bar{x}} \neq 0$ in $Z(G/D_{m,\hat{Z}_p}(G))$ which contradicts $x \in \bigcap_n \Delta_Z^n(G) \subseteq \Delta_Z(G, D_{m,\hat{Z}_p}(G))$.

Hence $\bigcap_m \Delta_Z^m(G) \subseteq \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G))$ for all p.

Conversely, if $x \in \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G))$ for all p, then $x \in \Delta_Z(G) \cap \Delta_{\hat{Z}_p}^n(G)$ for all p and all n, and Lemma 2.12 says that $x \in \bigcap_n \Delta_Z^n(G)$. Hence $\bigcap_n \Delta_Z^n(G) = \bigcap_p \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G))$.

Now $\bigcap_n D_{n,\hat{Z}_p}(G)$ is finite and p-torsion-free by assumption (where $p \in \beta$). If $\bigcap_n D_{n,\hat{Z}_p}(G) = 1$, then Theorem 1.2 and Lemma 2.4 say that $G \in RF_p$ and $\bigcap_n \Delta_Z^n(G) = 0$ by Theorem 2.1. If $\bigcap_n D_{n,\hat{Z}_p}(G) \neq 1$ for some $p \in \beta$, then choose $h \in \bigcap_n D_{n,\hat{Z}_p}(G)$ of prime order $q \neq p$. The torsion elements of G form a finite normal subgroup T where T equals $\bigcap_n D_{n,\hat{Z}_p}(G) \cdot \bigcap_n D_{n,\hat{Z}_q}(G)$. T is also nilpotent since $\bigcap_{p \in \beta} \bigcap_n D_{n,\hat{Z}_p}(G) = 1$ by the hypotheses, so $\bigcap_n D_{n,Z}(G) = 1$ and T is finite.

Let $T = S_{p_1} \times \ldots \times S_{p_n}$, the S_{p_k} being the p_k-Sylow subgroups of T. Note that for each p_k, $\bigcap_n D_{n,\hat{Z}_{p_k}}(G) = \prod_{j \neq k} S_{p_j}$. Therefore, we may apply Lemma 2.10 to conclude that $\bigcap_p \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G)) = \sum_{j \neq k} \Delta_Z(G, S_{p_k})\Delta_Z(G, S_{p_j})$. Hence (see two paragraphs back) we conclude that $\bigcap_n \Delta_Z^n(G) = \bigcap_p \Delta_Z(G, \bigcap_n D_{n,\hat{Z}_p}(G)) = \sum_{j \neq k} \Delta_Z(G, S_{p_k})\Delta_Z(G, S_{p_j})$. By Lemma 2.11, there exists $i \in \Delta_Z(T)$ such that $\bigcap_n \Delta_Z^n(T) = \sum_{j \neq k} \Delta_Z(T, S_{p_k})\Delta_Z(T, S_{p_j})$ annihilates $1 - i$. Since $S_{p_j} \triangleleft G$ for all j, $1 - i$ will also annihilate $\bigcap_n \Delta_Z^n(G)$. This completes the proof.

In the next case, we consider $R = \hat{Z}_p$ and G to be finitely generated, and obtain necessary and sufficient conditions for such an i to exist.

<u>Proposition 2.14</u>. Let G be finitely generated. Then the following are equivalent:

(i) there exists $i \in \Delta_{\hat{Z}_p}(G)$ such that $[\bigcap_n \Delta_{\hat{Z}_p}^n(G)](1 - i) = 0$.

(ii) $\bigcap_n D_{n,\hat{Z}_p}(G)$ is finite and of order relatively prime to p.

<u>Proof</u>. Assume first that (i) holds. If $\bigcap_n \Delta_{\hat{Z}_p}^n(G) = 0$, then $\bigcap_n D_{n,\hat{Z}_p}(G) = 1$. If $\bigcap_n \Delta_{\hat{Z}_p}^n(G) \neq 0$, let $g \in \bigcap_n D_{n,\hat{Z}_p}(G)$. Then $(g - 1)(1 - i) = 0$ so $\bigcap_n D_{n,\hat{Z}_p}(G)$ is

finite and $1 - i \in (\sum_{g \in \cap_n D_{n,\hat{Z}_p}(G)} g)\hat{Z}_p G$. Hence the content 1 of $1 - i$ is divisible by $|\cap_n D_{n,\hat{Z}_p}(G)|$, and we conclude that $\cap_n D_{n,\hat{Z}_p}(G)$ is of order relatively prime to p.

Now assume (ii) holds. It is easy to see that $\cap_n D_{n,\hat{Z}_p}(G/\cap_n D_{n,\hat{Z}_p}(G)) = 1$ which implies that $\cap_n \Delta^n_{\hat{Z}_p}(G/\cap_n D_{n,\hat{Z}_p}(G)) = 0$ by Theorem 2.5. Hence $\cap_n \Delta^n_{\hat{Z}_p}(G) \subseteq \Delta_{\hat{Z}_p}(G,\cap_n D_{n,\hat{Z}_p}(G))$. Since $\cap_n D_{n,\hat{Z}_p}(G)$ is finite and of order relatively prime to p, any $g \in \cap_n D_{n,\hat{Z}_p}(G)$ satisifies $(g - 1)(\frac{1}{|\cap_n D_{n,\hat{Z}_p}(G)|}\sum_{g \in \cap_n D_{n,\hat{Z}_p}(G)} g) = 0$ and the theorem holds with $i = 1 - \frac{1}{|\cap_n D_{n,\hat{Z}_p}(G)|}\sum_{g \in \cap_n D_{n,\hat{Z}_p}(G)} g$.

We also have a result in this area for finite groups G and commutative integral domains R.

Proposition 2.15. Let G be finite and R a commutative integral domain with 1. Then the following are equivalent:

(i) There exists $i \in \Delta_R(G)$ such that $[\cap_n \Delta^n_R(G)](1 - i) = 0$.

(ii) $\cap_n D_{n,R}(G)$ has order invertible in R and, if there exists $g \in G$ of order p, then either p is a unit in R or $\cap_n p^n R = 0$.

Proof. Assume (i) holds. If $g \in \bigcap_n D_{n,R}(G)$, then $(g - 1)(1 - i) = 0$ so, as before, $|\bigcap_n D_{n,R}(G)|$ is invertible in R. If there exists $g \in G$ of order p and $r \in \bigcap_n p^n R$, then $r(g - 1)(1 - i) = 0$ since $r(g - 1) \in \bigcap_n \Delta_R^n(G)$. Because R is an integral domain, $(g - 1)(1 - i) = 0$ which says, as usual, that p is invertible in R.

Note that the restriction that G be finite was not used in this section of the proof.

Now assume (ii) holds. Since G is finite, $\bigcap_n D_{n,R}(G) = D_{m,R}(G)$ for some m. Let $H = G/D_{m,R}(G)$. Then H is finite and nilpotent. By observing that the proof of Lemma 2.11 goes through for R a commutative integral domain, we can find $\bar{i} \in \Delta_R(H)$ such that $[\bigcap_n \Delta_R^n(H)](1 - \bar{i}) = 0$. Consequently, $[\bigcap_n \Delta_R^n(G)](1 - i) \in \Delta_R(G, D_{m,R}(G))$ and $i \in \Delta_R(G)$. Since $D_{m,R}(G) = \bigcap_n D_{n,R}(G)$ has order invertible in R, we see as in the last part of the proof of Proposition 2.14 that there exists $j \in \Delta_R(G)$ with $[\bigcap_n \Delta_R^n(G)](1 - i)(1 - j) = 0$. This completes the proof.

Note. The study of powers of the augmentation ideal can be extended to transfinite powers where $\Delta_R^\alpha(G)$ is defined to be $\Delta_R^{\alpha-1}(G)\Delta_R(G)$ whenever $\alpha - 1$ makes sense and $\bigcap_{\lambda<\alpha} \Delta_R^\lambda(G)$ otherwise. Gruenberg and Roseblade [16] called the smallest ordinal ω with $\Delta_R^\omega(G) = \Delta_R^{\omega+1}(G)$ the augmentation terminal of the group ring, and they studied this for locally finite groups. Note that all theorems in

this section which give conditions when $[\bigcap_n \Delta_R^n(G)](1 - i) = 0$ for some $i \in \Delta_R(G)$ automatically give sufficient conditions for the augmentation terminal of the group ring to be $\leq \omega$.

3. $G \cap (1 + \mathrm{Ann} \bigcap_n \Delta_R^n(G))$

We would like to very briefly discuss another problem which can be tackled using Theorem 1.2. The problem is to determine when one can find $g \in G$, $g \neq 1$, such that $(g - 1)[\bigcap_n \Delta_R^n(G)] = 0$. This is related to the work of Sandling ([34],[35]), who was concerned with $\{g \mid I^j(g - 1)I^k = 0 \text{ whenever } j + k = n\}$ for each n, and called these subgroups dual to dimension subgroups. We prove the following:

Proposition 2.16. Let G be a finitely generated torsion group with $|G| > 2$. Let R be a commutative ring of characteristic $\neq 2$. Then the following are equivalent:

(i) there exists $g \in G$, $g \neq 1$, such that $(g - 1)[\bigcap_n \Delta_R^n(G)] = 0$

(ii) $\bigcap_n \Delta_R^n(G) = 0$.

Proof. Assume $g \neq 1$ is such that $(g - 1)[\bigcap_n \Delta_R^n(G)] = 0$. Let a,b be of coprime order in G (if possible). Then $(a - 1)(b - 1) \in \bigcap_n \Delta_R^n(G)$ so $(g - 1)(a - 1)(b - 1) = 0$. Therefore, $gab - ga - gb - ab + a + b + g - 1 = 0$.

Since $\operatorname{char} R \neq 2$, the only possibility is $gab = 1$, $ga = b$, $gb = a$ and $ab = g$, which yields $gb^2 = ab = g$ and $b^2 = 1$. Similarly, we conclude from $(g - 1)(b - 1)(a - 1) = 0$ that $a^2 = 1$, and this contradicts a and b being of coprime order. Hence G is a p-group.

Clearly, $\bigcap_n D_{n,R}(G) = 1$ since $x \in \bigcap_n D_{n,R}(G)$ implies $(g - 1)(x - 1) = 0$ and, since $\operatorname{char} R \neq 2$, $x = 1$.

Also, G being a p-group implies $\bigcap_n p^n R = 0$ since $r \in \bigcap_n p^n R$ and $x \in G$ implies $r(x - 1) \in \bigcap_n \Delta_R^n(G)$ and $r(g - 1)(x - 1) = 0$. Since $|G| \neq 2$, this implies $r = 0$.

Now $G/D_{n,R}(G)$ is a finitely generated, nilpotent p-group, hence, by Lemma 2.4, is in $\mathcal{RF}_p$. Theorem 2.1 says that $\bigcap_n \Delta_R^n(G) = 0$.

Rings of characteristic $\neq 0$ but not of characteristic $2r$ where r is odd can be broken into prime power characteristic parts and handled by the following:

Proposition 2.17. Let $\operatorname{char} R = p^e$ where $p^e \neq 2$. Let G be finitely generated. Then the following are equivalent:

(i) there exists g, $g \neq 1$, such that

$$(g - 1)[\bigcap_n \Delta_R^n(G)] = 0$$

(ii) $\bigcap_n \Delta_R^n(G) = 0$.

Proof. As usual, $(g - 1)[\bigcap_n \Delta_R^n(G)] = 0$ implies $\bigcap_n D_{n,R}(G) = 1$. Also $G/D_{n,R}(G)$ is nilpotent, finitely generated and (since every prime but p is invertible in R) contains only torsion elements of p-power order. Hence, by Lemma 2.4, $G \in \mathcal{RF}_p$ and Theorem 2.1 completes the result.

Note. An identical argument to Proposition 2.17 works for any ring in which every prime but p is invertible and for which $\bigcap_n p^n R = 0$, for example $\hat{Z}_p$.

CHAPTER 3

Units and Radicals of Certain Group Rings

In this chapter, we study the units and Jacobson radical for group rings of certain very special groups over fairly general rings of coefficients. In the case of units, we are primarily concerned with right-ordered groups while, in the case of the radical, we restrict ourselves even further to infinite cyclic groups. This may seem like a rather specialized topic, but we approach it for two reasons. Firstly, it is interesting to see how the techniques and results of Coleman and Enochs [9] and of Amitsur [2] for polynomial rings apply to group rings. Secondly, some of the results in this chapter are required in the study of isomorphism problems in chapter 4.

1. Units

The main result of this section is the determination of the invertible elements of RG when G is a right-ordered group and R has no nontrivial nilpotent elements. We then obtain several corollaries to this result, and close the section with a result on the automorphisms of $R\langle x\rangle$ when R is commutative semiprime and $\langle x\rangle$ is an infinite cyclic group.

It is unreasonable to expect a complete determination of the invertible elements of RG when G is right-ordered because, in general, it is not known exactly which elements in a polynomial ring $R[x]$ are units. Some results on the latter problem have been obtained by Coleman and Enochs [9].

Let $U_{rt}(RG)$ denote the right-invertible elements of RG.

Theorem 3.1. Let G be right-ordered. Then the following are equivalent:

(i) $U_{rt}(RG) = \{\sum \alpha_g g \mid$ there exist $\beta_g \in R$ with $\sum_g \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h = 0$ whenever $h \neq g^{-1}\}$.

(ii) R has no non-zero nilpotent elements.

Proof. Assume (i) holds and let $n \in R$ be nilpotent. Then $1 + ng$, for any $g \neq 1$ in G, is a right-invertible element of RG, but does not satisfy the conditions of (i).

Conversely, assume (ii) holds and let $yz = 1$ where $y = \sum_{i=1}^{r} \alpha_i g_i$, $z = \sum_{i=1}^{s} \beta_i h_i$. We will show that $\alpha_i \beta_j = 0$ whenever $h_j \neq g_i^{-1}$. The other statement follows immediately. Suppose that $g_1 < \ldots < g_r$ and $h_1 < \ldots < h_s$. For any fixed j, we know that $g_1 h_j < \ldots < g_r h_j$. Choose j_1,

$1 \le j_1 \le s$, with $g_r h_{j_1}$ maximal in $\{g_r h_j\}_j$. If $g_r h_{j_1} > 1$, then we notice from the choice of j_1 that the term $g_r h_{j_1}$ does not occur again in the set of products $g_i h_k$. Hence we conclude that $\alpha_r \beta_{j_1} = 0$. Assume that we know that $\alpha_\ell \beta_m = 0$ whenever $g_\ell h_m > g_{i_1} h_{k_1} = g_{i_2} h_{k_2} = \ldots = g_{i_p} h_{k_p} > 1$ (the $g_{i_s} h_{k_s}$ being a complete list of products equal to $g_{i_1} h_{k_1}$). We see that $\alpha_{i_1} \beta_{k_1} + \ldots + \alpha_{i_p} \beta_{k_p} = 0$ and we may suppose that $i_1 < i_2 < \ldots < i_p$. Multiplying on the right by α_{i_p}, we obtain $\alpha_{i_1} \beta_{k_1} \alpha_{i_p} + \ldots + \alpha_{i_p} \beta_{k_p} \alpha_{i_p} = 0$. For $t < p$, $g_{i_p} h_{k_t} > g_{i_t} h_{k_t}$. Hence, by induction, $\alpha_{i_p} \beta_{k_t} = 0$. This implies, by assumption (ii), that $\beta_{k_t} \alpha_{i_p} = 0$ since $(\beta_{k_t} \alpha_{i_p})^2 = 0$. Hence, we have $\alpha_{i_p} \beta_{k_p} \alpha_{i_p} = 0$ and, again using (ii), $\alpha_{i_p} \beta_{k_p} = 0$. Working back, we obtain $\alpha_{i_t} \beta_{k_t} = 0$ for $1 \le t \le p$. Therefore, we have shown that $\alpha_i \beta_j = 0$ whenever $g_i h_j > 1$.

An identical argument to that given above, starting with $g_1 h_{j_1}$ minimal in $\{g_1 h_j\}_j$, shows that $\alpha_i \beta_j = 0$ whenever $g_i h_j < 1$. This completes the proof.

Note. Although the condition that G be right-ordered is one sided, the above proof also gives the structure of all left and 2-sided units of RG when G is right-ordered and R has no non-zero nilpotent elements.

Corollary 3.2 (see [40]). Let G be right-ordered and let R have no idempotents $\neq 0,1$. Then the following are equivalent:

(i) $U_{rt}(RG) = \{ug \mid u$ is a right unit of $R\}$.

(ii) R has no non-zero nilpotent elements.

Proof. (i) $\Rightarrow$ (ii) is trivial. Assume (ii) holds and let $\sum \alpha_g g$ be a right unit of RG with right inverse $\sum \beta_g g$. Theorem 3.1 says that $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h = 0$ for $h \neq g^{-1}$.

Note that $\sum \alpha_g \beta_{g^{-1}} = 1$ implies that $(\sum \alpha_g \beta_{g^{-1}})\alpha_h = \alpha_h$ for all h. But for $h \neq g$, $\alpha_h \beta_{g^{-1}} = \beta_{g^{-1}} \alpha_h = 0$ by (ii). Hence we obtain $\alpha_g \beta_{g^{-1}} \alpha_g = \alpha_g$ for all g. If $\alpha_g \neq 0$, we conclude $\alpha_g \beta_{g^{-1}} = 1$ since R has no idempotents $\neq 0,1$. Also, we have $\beta_{g^{-1}} \alpha_g = 1$. This implies that α_g is a unit. Since $\alpha_g \beta_h = 0$ for $h \neq g^{-1}$, we have $\beta_h = 0$ for $h \neq g^{-1}$ and the result follows.

Note. Since R has no idempotents $\neq 0,1$, u is right-invertible in R if, and only if, it is left-invertible in R.

Corollary 3.3. Let G be right-ordered. If the nilpotent elements of R form an ideal, then a right unit in RG has the form $\sum \alpha_g g$ such that there exist $\beta_g \in R$ with $\sum \alpha_g \beta_{g^{-1}} = 1$, and $\alpha_g \beta_h$ is nilpotent whenever $h \neq g^{-1}$.

Note. If R is a ring with 1 satisfying the condition $xy = 0$ implies $yx = 0$, then the nilpotent elements of R form an ideal N, so Corollary 3.3 applies. In this case, as is shown in [9], the ideal NG of RG is nil,

and the converse to Corollary 3.3 follows also.

For abelian groups, we can use the above results to generalize a result of Sehgal [40] to commutative integral domains. First we note several lemmas. The first specializes a result of Passman:

Lemma 3.4 [29]. Let R be a commutative integral domain of characteristic zero and let G be abelian. Then RG has no non-trivial nilpotent elements.

The second lemma is a special case of a result of Coleman:

Lemma 3.5 [7]. Let R be a commutative integral domain and let G be abelian such that if there exists $g \in G$ of order p, then p is not invertible in R. Then RG has no nontrivial idempotent elements.

Remark. Coleman only proved this for finite groups G. To pass to the infinite case, we require the observation that central idempotents in a group ring have finite support group, a simple proof of which may be found in Burns [6].

We now state our result:

Theorem 3.6 (see [40]). Let R be a commutative integral domain of characteristic zero. Let G be an abelian group such that if there exists $g \in G$ of order p, then p is not invertible in R. Then $U(RG) = \{ug \mid g \in G$ and u

is a unit of RT where T is the torsion subgroup of $G\}$.

Proof. We may assume G is finitely generated, say $G = T \times \langle x_1\rangle \times \ldots \times \langle x_s\rangle$ where T is torsion and the x_i are of infinite order. By Lemmas 3.4 and 3.5, $R(T \times \langle x_1\rangle \times \ldots \times \langle x_{s-1}\rangle)$ has no non-trivial nilpotent or idempotent elements. By Corollary 3.2, any unit in RG is of the form ux_s^i where u is a unit of $R(T \times \langle x_1\rangle \times \ldots \times \langle x_{s-1}\rangle)$. The result follows by induction on the number of infinite cyclic factors s.

We note that the following simple extension also holds:

Corollary 3.7 (see [40]). Let R be a commutative integral domain of characteristic zero and let G be the direct product of an abelian group A with a right-ordered group 0, such that if there exists $g \in G$ of order p, then p is not invertible in R. Then $U(RG) = \{ug \mid g \in G,\ u$ a unit of RT where T is the torsion subgroup of $G\}$.

Proof. $G = A \times 0$. $RG = (RA)0$ and Lemmas 3.4 and 3.5 say RA has no nontrivial idempotent or nilpotent elements. By Corollary 3.2 and Theorem 3.6, the result follows.

At this stage, we mention one result concerning automorphisms of the group ring $R\langle x\rangle$ where $\langle x\rangle$ is an infinite cyclic group. The automorphisms of the polynomial

ring $R[x]$ have been completely determined; by Gilmer [13] in the commutative case and by Coleman and Enochs [9] in the noncommutative case. The fact that x becomes a unit complicates matters greatly when one passes to $R\langle x\rangle$. However, we have the following:

<u>Proposition 3.8</u>. Let R be commutative, semiprime with 1. Then $x \to \sum_{-s}^{t} a_i x^i$ induces on R-algebra automorphism of $R\langle x\rangle$ if, and only if, $\sum_{-s}^{t} a_i x^i$ is a unit and $a_i = 0$ for $i \neq 1,-1$.

<u>Proof</u>. Assume first that $x \to \sum_{-s}^{t} a_i x^i$ induces an R-algebra automorphism of $R\langle x\rangle$. Clearly $\sum_{-s}^{t} a_i x^i$ is a unit and there exist $c_j \in R$ such that $x = \sum c_j (\sum_{-s}^{t} a_i x^i)^j$. Note also that $(\sum_{-s}^{t} a_i x^i)^{-1} = \sum k_i x^i$ where $\sum a_i k_{-i} = 1$ and $a_i k_j = 0$ for $j \neq -i$ by Theorem 3.1. Furthermore, observe that if P is a prime ideal of R, Corollary 3.2 tells us that exactly one a_i and exactly one k_i do not lie in P (by going to R/P). Consequently, if $j \neq i$, $a_i a_j \in \cap P = 0$ and, similarly, $k_i k_j = 0$. From $x = \sum c_j (\sum_{-s}^{t} a_i x^i)^j$, we obtain $1 = c_1 a_1 + c_{-1} k_1$ by comparing coefficients of x and using the above observations.

Hence, a_1 and k_1 lie in no common prime ideals of R. However $\sum a_i k_{-i} = 1$ and $a_i k_j = 0$ for $j \neq -i$ implies that $k_1 a_{-1} k_1 = k_1$. Hence a_1 and a_{-1} lie in no common prime ideals of R. If $a_i \neq 0$ for $i \neq 1,-1$, choose P so that $a_i \notin P$. By the previous paragraph, a_1 and a_{-1} both lie in P, and this is not possible. Hence $a_i = 0$ for $i \neq 1,-1$.

Conversely, assume $\sum_{-s}^{t} a_i x^i$ is a unit and $a_i = 0$ for $i \neq 1,-1$. Consider the R-algebra map defined by $x \to a_{-1}x^{-1} + a_1 x$. This map is onto, since if $(a_{-1}x^{-1} + a_1 x)^{-1} = k_{-1}x^{-1} + k_1 x$, then $x = k_{-1}(a_{-1}x^{-1} + a_1 x) + a_{-1}(k_{-1}x^{-1} + k_1 x)$.

Suppose $\sum c_j (a_{-1}x^{-1} + a_1 x)^j = 0$. Consider $R/P\langle x\rangle$ where P is a prime ideal of R. Then $\sum \overline{c_j}(\overline{a_{-1}}x^{-1} + \overline{a_1}x)^j = 0$ and we know as before that either a_1 or $a_{-1} \in P$. Say $a_{-1} \in P$, $a_1 \notin P$. Then $\sum \overline{c_j}\,\overline{a_1}^j x^j = 0$ and hence $\overline{c_j}\,\overline{a_1}^j = 0$ and $c_j \in P$. This is true for all j and all P. Hence $c_j \in \cap P = 0$ and the map is $1-1$.

This completes the proof.

2. Jacobson Radical

We now proceed to discuss the problem of finding the Jacobson radical $J(R\langle x\rangle)$ of the group ring of an infinite cyclic group. The result for polynomial rings is

the following theorem of Amitsur:

Theorem 3.9 [2]. $J(R[x]) = N[x]$ where $N = J(R[x]) \cap R$ is a nil ideal of R.

We have as yet been unable to find a similar theorem for group rings. In particular, it would be nice to know whether $J(R\langle x\rangle) = (J(R\langle x\rangle) \cap R)\langle x\rangle$. Our main result in this section (Theorem 3.13) consists of finding an alternate statement of this assertion. Some of these results will be required in chapter 4.

Proposition 3.10. Let N be as in Theorem 3.9. Then $J(R\langle x\rangle) \subseteq N\langle x\rangle$.

Proof. Let $a = \sum a_i x^i \in J(R\langle x\rangle)$. We may assume $a = a_1 x + \ldots + a_n x^n$. Now a has a right quasi-inverse $b \in R\langle x\rangle$ with $a + b + ab = 0$. We see that $b \in R[x]$ by the choice of a. More generally, the ideal of $R[x]$ generated by a is right-quasi-regular, and hence is contained in $J(R[x]) = N[x]$ by Theorem 3.9. The result follows immediately.

We can conclude that $J(R\langle x\rangle) = N\langle x\rangle$ whenever N is nilpotent, since then $N\langle x\rangle$ will be nil (obviously $J(R\langle x\rangle) = N\langle x\rangle$ could be true much more generally). For example, if R has A.C.C. on left ideals, then every nil ideal of R is nilpotent and $J(R\langle x\rangle) = N\langle x\rangle$.

We also observe that Proposition 3.10 yields $J(R\langle x\rangle) \subseteq J(R)\langle x\rangle$ as a corollary. A necessary condition for the converse to hold is that $J(R)$ be nil since $J(R) \subseteq N$ would be required, but it is not clear that this is sufficient. In certain cases, however, this does hold.

<u>Proposition 3.11</u>. If R satisfies the condition that every nilpotent element is strongly nilpotent or satisfies a polynomial identity or is an algebra over a non-denumerable field, then $J(R)$ nil implies $J(R\langle x\rangle) = J(R)\langle x\rangle$.

<u>Proof</u>. If every nilpotent element is strongly nilpotent, $\beta(R\langle x\rangle) \subseteq J(R\langle x\rangle) \subseteq N\langle x\rangle \subseteq \beta(R)\langle x\rangle \subseteq \beta(R\langle x\rangle)$ by a result of Connell [10] where $\beta(R)$ is the prime radical of R, so equality holds and $J(R\langle x\rangle) = \beta(R)\langle x\rangle$. The hypotheses imply $J(R) = \beta(R)$ so the result follows.

The polynomial identity case works out since nil ideals are locally nilpotent, and the case of an algebra over a nondenumerable field is found in Amitsur [2].

<u>Note</u>. The problem of determining when $J(RG) = J(R)G$ has been considered previously by Coleman [8].

Analogous to the polynomial ring case, we have:

<u>Proposition 3.12</u>. $J(R\langle x\rangle) \cap R$ is a nil ideal of R.

Proof. Let $r \in J(R\langle x\rangle) \cap R$. Then r^2x has a right-quasi-inverse $q(x) = \sum_{-s}^{t} a_i x^i$. $r^2x + q(x) + r^2xq(x) = 0$, so, by iteration, $q(x) = -r^2x + r^4x^2 + \ldots + (-1)^n r^{2n}x^n + (-1)^n r^{2n}x^n q(x)$. Choose $n > s + t + 1$ and the coefficient of $x^{t+1} = (-1)^{t+1}r^{2(t+1)} = 0$, so r is nilpotent.

Theorem 3.13. The following statements are equivalent.

(i) For every ring R, $J(R\langle x\rangle) = (J(R\langle x\rangle) \cap R)\langle x\rangle$.

(ii) For every ring R, $J(R\langle x\rangle) \cap R = 0$ implies $J(R\langle x\rangle) = 0$.

Proof. (i) implies (ii) is immediate. Hence assume (ii) holds and say $J(R\langle x\rangle) \neq 0$. Let $N = J(R\langle x\rangle) \cap R \neq 0$ by (ii). If $J(\frac{R}{N}\langle x\rangle) \neq 0$, then (ii) implies $J(\frac{R}{N}\langle x\rangle) \cap R/N \neq 0$ and hence there exists $0 \neq \bar{r} \in J(\frac{R}{N}\langle x\rangle) = \frac{J(R\langle x\rangle)}{N\langle x\rangle}$ (note $N\langle x\rangle \subseteq J(R\langle x\rangle)$). Hence $r \in J(R\langle x\rangle) + N\langle x\rangle \subseteq J(R\langle x\rangle)$ and $\bar{r} = 0$, which is a contradiction. Therefore $J(\frac{R}{N}\langle x\rangle) = 0$ and $J(R\langle x\rangle) \subseteq N\langle x\rangle$. Hence $J(R\langle x\rangle) = N\langle x\rangle$ and (i) holds.

CHAPTER 4

Isomorphism of Certain Group Rings

In this chapter, we consider the extent to which a group ring RG determines its ring of coefficients R. That is, if $RG \simeq SG$ what can we conclude about R and S? We shall be dealing solely with the case where G is an infinite cyclic group. Our main result shows that R is determined by $R\langle x\rangle$ whenever R is in a certain class of rings large enough to contain all rings with perfect centres. We also show that if R contains no nontrivial nilpotent or idempotent elements, then $R\langle x\rangle \simeq S\langle x\rangle$ implies R can be embedded in S and S can be embedded in R.

The analogous problem for polynomial rings is whether or not $R[x] \simeq S[x]$ implies $R \simeq S$. This question was raised by Coleman and Enochs [9], who proved the result for several cases including perfect rings and certain integral domains. Other positive results on this problem have been obtained by Abhyankar, Eakin and Heinzer [1] and by Jacobson [20]. A negative result has recently been obtained by Hochster [19], who found two non-isomorphic integral domains R,S with $R[x] \simeq S[x]$. We will also show that Hochster's counterexample does not apply to the problem $R\langle x\rangle \simeq S\langle x\rangle$.

Before we prove our main result, we give several preliminary lemmas. First recall from before that $(R_1 \oplus R_2)G \simeq R_1G \oplus R_2G$ under the map $\sum_i (r_i,s_i)g_i \to (\sum r_ig_i, \sum s_ig_i)$. We will identify these two isomorphic rings whenever it is convenient to do so.

Now we show that infinite cyclic group rings over fields determine their rings of coefficients.

Lemma 4.1. Let F,K be fields such that $\sigma{:}F\langle x\rangle \to K\langle x\rangle$ is an isomorphism. Then $\sigma(F) = K$.

Proof. Let $f \neq 0,-1$ be an element of F. Since $\sigma(f)$ is a unit of $K\langle x\rangle$, we have by Corollary 3.2 that $\sigma(f) = kx^i$ for some $k \in K$. But $\sigma(1 + f) = 1 + kx^i$ is also a unit. Corollary 3.2 yields $i = 0$ and hence $\sigma(F) \subseteq K$. By using σ^{-1}, we conclude that $\sigma(F) = K$.

Lemma 4.2. Let R and S be finite direct sums of fields. Then $R\langle x\rangle \overset{\sigma}{\simeq} S\langle x\rangle$ implies $\sigma(R) = S$.

Proof. Let $R = F_1 \oplus \ldots \oplus F_n$, $S = K_1 \oplus \ldots \oplus K_m$, direct sums of fields F_i and K_j. With the usual identification, $R\langle x\rangle = F_1\langle x\rangle \oplus \ldots \oplus F_n\langle x\rangle$ and $S\langle x\rangle = K_1\langle x\rangle \oplus \ldots \oplus K_m\langle x\rangle$. The only primitive idempotents in $R\langle x\rangle$ are $\ell_i = (0,\ldots 0,1,0,\ldots 0)$ and similarly for $S\langle x\rangle$. Hence, for each i, $\sigma(F_i\langle x\rangle) = K_j\langle x\rangle$ for some j and $m = n$. By Lemma 4.1, $\sigma(F_i) = K_j$ and it follows that $\sigma(R) = S$.

Definition. A ring R is called left perfect if $R/J(R)$ is left Artinian and $J(R)$ is left T-nilpotent.

Lemma 4.3. Let R be a perfect commutative ring. Then $J(R\langle x\rangle) = J(R)\langle x\rangle$.

Proof. Since R is perfect, $J(R)$ is T-nilpotent. In addition, R is commutative, so $J(R)$ is nil. Hence $J(R)\langle x\rangle \subseteq J(R\langle x\rangle)$. By Proposition 3.10, this implies $J(R\langle x\rangle) = J(R)\langle x\rangle$.

We now proceed to the main result. Recall that a commutative ring R is called semi-perfect if $R/J(R)$ is artinian and if idempotents can be lifted modulo $J(R)$.

Theorem 4.4. Let R_i be a ring with 1 $(i = 1,2)$. Suppose that (i) Z_i, the centre of R_i, is semi-perfect for $i = 1,2$ and (ii) $J(Z_i\langle x\rangle) = J(Z_i)\langle x\rangle$ for $i = 1,2$. Then $R_1\langle x\rangle \simeq R_2\langle x\rangle$ implies $R_1 \simeq R_2$.

Proof. Let $\sigma: R_1\langle x\rangle \to R_2\langle x\rangle$ be the isomorphism. Then by restriction we have $\sigma: Z_1\langle x\rangle \to Z_2\langle x\rangle$. Due to the second hypothesis, we have an induced isomorphism $\bar{\sigma}: \frac{Z_1}{J(Z_1)}\langle x\rangle \to \frac{Z_2}{J(Z_2)}\langle x\rangle$.

By Lemma 4.2, it follows that $\bar{\sigma}\left(\frac{Z_1}{J(Z_1)}\right) = \frac{Z_2}{J(Z_2)}$ since Z_i is semi-perfect. We identify $\frac{Z_2}{J(Z_2)}\langle x\rangle$ with $\overline{\ell_1}\frac{Z_2}{J(Z_2)}\langle x\rangle \oplus \dots \oplus \overline{\ell_n}\frac{Z_2}{J(Z_2)}\langle x\rangle$ where each $\overline{\ell_i}$ is a

primitive idempotent and each $\overline{\ell_i}\,\frac{Z_2}{J(Z_2)}$ is a field F_i. Since $\overline{\sigma}\,(\frac{Z_1}{J(Z_1)})$ and $\overline{\sigma}(x)$ together must generate $\frac{Z_2}{J(Z_2)}\langle x\rangle$, we see that $\overline{\sigma}(x) = (f_1x^{i_1},\ldots f_nx^{i_n})$ where $i_j = \pm 1$ and $0 \neq f_j \in F_j$ for all j. Since Z_2 is semi-perfect, we can lift the idempotents $\overline{\ell_i}$ of $\frac{Z_2}{J(Z_2)}$ to primitive orthogonal idempotents ℓ_i in Z_2 such that $1 = \ell_1 + \ell_2 + \ldots + \ell_n$. Then we obtain $R_2\langle x\rangle = \ell_1R_2\langle x\rangle \oplus \ldots \oplus \ell_nR_2\langle x\rangle$.

Define on R_2-algebra automorphism $\beta: R_2\langle x\rangle \to R_2\langle x\rangle$ by $\beta(x) = (x^{i_1},\ldots x^{i_n})$. It is not difficult to check that β is indeed an automorphism and therefore induces an automorphism $\overline{\beta}$ of $\frac{Z_2}{J(Z_2)}\langle x\rangle$. Notice that $\overline{\beta}\,\overline{\sigma}(x) = (f_1x,\ldots,f_nx) = (f_1,\ldots,f_n)x = ux$ where u is a unit in $\frac{Z_2}{J(Z_2)}$. Since $J(Z_2\langle x\rangle) = J(Z_2)\langle x\rangle$, we conclude from Proposition 3.10 that $J(Z_2)$ is a nil ideal. Hence, we see that $\beta\sigma(x) = u_1x + \sum_{i\neq 1} a_ix^i$ where u_1 is a unit in Z_2 and the a_i are nilpotent elements of Z_2.

We now claim that $R_2\langle\beta\sigma(x)\rangle = R_2\langle x\rangle$. We may assume that $u_1 = 1$. Note that $(\beta\sigma(x))^{-1} = x^{-1}(1 + \sum_{i\neq 0} a_{i+1}x^i)^{-1} = x^{-1}(1 - r + r^2 + \ldots + (-1)^sr^s)$ where $r = \sum_{i\neq 0} a_{i+1}x^i$ and $r^{s+1} = 0$. We proceed by

induction on the index of nilpotency of A, the ideal of R_2 generated by $\{a_i\}$. If this index is one, then $\beta\sigma(x) = x$ and we are finished. Now we can suppose that this index is greater than one. Observing that $(\beta\sigma(x))^i = x^i + \sum_j b_j x^j$, $b_j \in A$, we obtain $\beta\sigma(x) - \sum_{i\neq 1} a_i(\beta\sigma(x))^i =$ $x - \sum a_i b_j x^j$ where a_i, $b_j \in A$. Since A is nilpotent, $\beta\sigma(x) - \sum_{i\neq 1} a_i(\beta\sigma(x))^i = vx - \sum_{j\neq 1} a_i b_j x^j$ where v is a unit in Z_2 and $a_i b_j \in A^2$ which is of smaller index of nilpotency than A. Hence, $R_2\langle\beta\sigma(x) - \sum_{i\neq 1} a_i(\beta\sigma(x))^i\rangle =$ $R_2\langle x\rangle$ and, therefore, $R_2\langle\beta\sigma(x)\rangle = R_2\langle x\rangle$.

Consequently, we have proved that the R_2-homomorphism $\alpha : R_2\langle x\rangle \to R_2\langle x\rangle$ defined by $\alpha(x) = \beta\sigma(x)$ is an epimorphism.

To see that α is $1 - 1$, assume that $\sum c_i(\beta\sigma(x))^i = 0$. In $\frac{R}{A}\langle x\rangle$, we obtain $\sum \overline{c_i}\ \overline{\beta\sigma(x)}^{\,i} = 0$. However, $\overline{\beta\sigma(x)} = \overline{u_1}x$. Hence we obtain $\overline{c_i}\ \overline{u_1}^{\,i} = 0$ and, therefore, $c_i \in A$. Assume $c_i \in A^k$ for all i. Then $\sum c_i(\beta\sigma(x))^i \in A^k\langle x\rangle$. In $\frac{A^k\langle x\rangle}{A^{k+1}\langle x\rangle}$, $\overline{\sum c_i(\beta\sigma(x))^i} = 0$ which implies again that $c_i u_1^{\,i} \in A^{k+1}$ and $c_i \in A^{k+1}$. Since A is nilpotent, we conclude that $c_i = 0$ for each i and, hence, α is $1 - 1$.

Now we have a ring isomorphism $\alpha^{-1}\beta\sigma : R_1\langle x\rangle \to$ $R_2\langle x\rangle$ such that $(\alpha^{-1}\beta\sigma)(x) = x$ and, therefore,

$(\alpha^{-1}\beta\sigma)(\Delta_{R_1}(\langle x\rangle)) = \Delta_{R_2}(\langle x\rangle)$ where Δ denotes the augmentation ideal. Consequently, we obtain $R_1 \simeq \dfrac{R_1\langle x\rangle}{\Delta_{R_1}(\langle x\rangle)} \simeq \dfrac{R_2\langle x\rangle}{\Delta_{R_2}(\langle x\rangle)} \simeq R_2$. The proof is complete.

Corollary 4.5. Let R_1, R_2 be rings with perfect centres. Then $R_1\langle x\rangle \simeq R_2\langle x\rangle$ implies $R_1 \simeq R_2$.

Proof. Lemma 4.3 and Theorem 4.4.

Note that since a left Artinian ring is left perfect, we have shown the isomorphism theorem holds for rings with artinian centres.

Since we are considering this problem, we should notice the following:

Proposition 4.6. Let R_1 be a ring with 1 such that its centre Z_1 has no nontrivial idempotent or nilpotent elements. Suppose that all units of Z_1 are algebraic over the prime subring of Z_1. Then $R_1\langle x\rangle \overset{g}{\cong} R_2\langle x\rangle$ implies $R_1 \simeq R_2$.

Proof. We first remark that $Z_1\langle x\rangle$, and therefore also $Z_2\langle x\rangle$, has no nontrivial idempotent or nilpotent elements (Z_2 is the centre of R_2). It follows from Corollary 3.2 that the units of $Z_i\langle x\rangle$ are of the form ux^j with $u \in Z_i$ for $i = 1,2$. By the hypothesis, a unit in $Z_1\langle x\rangle$ is algebraic over the prime subring of Z_1 if, and only if,

it is in Z_1. Hence $\sigma(U(Z_1)) \subseteq U(Z_2)$ and $\sigma(x) = ux^{\ell}$, $\ell = \pm 1$ and $u \in Z_2$. Define $\tau: R_2\langle x\rangle \to R_2\langle x\rangle$ by

(i) $\tau(x) = u^{-\ell}x^{\ell}$

(ii) $\tau(\sum a_j x^j) = \sum a_j \tau(x)^j$.

Then τ is an R_2-algebra automorphism of $R_2\langle x\rangle$ and $\tau\sigma(x) = x$. Hence we see that $\tau\sigma(\Delta_{R_1}(\langle x\rangle)) = \Delta_{R_2}(\langle x\rangle)$, which implies that $R_1 \simeq R_2$.

We next observe the following interesting fact:

Proposition 4.7. Let R, S be commutative rings with no nontrivial idempotent or nilpotent elements. Then $R\langle x\rangle \overset{\theta}{\simeq} S\langle x\rangle$ implies R can be embedded in S and S can be embedded in R.

Proof. It suffices, by taking $\theta^{-1}(S\langle x\rangle)$, to consider the problem $R\langle x\rangle = T\langle ux^r\rangle$ where u is a unit in R and r is an integer (using Corollary 3.2).

First consider the case $r = 0$. Then x in $T\langle u\rangle$ implies that $x = vu^j$ for some j and some $v \in T$. v is also a unit of $R\langle x\rangle$, so must be of the form ax^s for some a in R. We conclude that $s = 1$ and $ax \in T$. Now $T\langle u\rangle = T\langle axu\rangle$ since ax is a unit in T and $R\langle x\rangle = R\langle aux\rangle$ since au is a unit in R. Therefore we have $T\langle aux\rangle = R\langle aux\rangle$ and $R \simeq T$ by factoring out augmentation ideals.

Hence we may assume $r \neq 0$. In that case, $R\langle x\rangle \supseteq R\langle x^r\rangle = R\langle ux^r\rangle$ and we have $R\langle ux^r\rangle \subseteq T\langle ux^r\rangle$. Therefore, $\Delta_R(\langle ux^r\rangle) = \langle ux^r - 1\rangle_{R\langle ux^r\rangle} \subseteq \langle ux^r - 1\rangle_{T\langle ux^r\rangle} \cap R\langle ux^r\rangle = \Delta_T(\langle ux^r\rangle) \cap R\langle ux^r\rangle$.

We claim that equality holds here. If $m \in \Delta_T(\langle ux^r\rangle) \cap R\langle ux^r\rangle$, then we have $m = t + w$ where $t \in R$ and $w \in \Delta_R(\langle ux^r\rangle)$. Hence $t \in \Delta_T(\langle ux^r\rangle) \cap R$ which means that $t = g(x)(ux^r - 1)$ for some $g(x) \in T\langle ux^r\rangle$. Since u is a unit in R, this is impossible unless $t = 0$.

Hence $\Delta_R(\langle ux^r\rangle) = \Delta_T(\langle ux^r\rangle) \cap R\langle ux^r\rangle$.

Therefore $R \simeq \dfrac{R\langle ux^r\rangle}{\Delta_R(\langle ux^r\rangle)}$ can be embedded in

$$T \simeq \frac{T\langle ux^r\rangle}{\Delta_T(\langle ux^r\rangle)} \quad \text{via} \quad R\langle ux^r\rangle \xrightarrow{i} T\langle ux^r\rangle \to \frac{T\langle ux^r\rangle}{\Delta_T(\langle ux^r\rangle)} \simeq T.$$

By arguing in the other direction, we get that T can be embedded in R also.

<u>Corollary 4.8</u>. Let R_1, R_2 be commutative rings with no non-trivial idempotents. Then $R_1\langle x\rangle \simeq R_2\langle x\rangle$ implies $R_1/\beta(R_1)$ can be embedded in $R_2/\beta(R_2)$ and conversely, where $\beta(R)$ is the prime radical of R.

We would like to close this chapter by showing that Hochster's [19] counterexample to the polynomial ring case is not a counterexample to the $R\langle x\rangle \simeq S\langle x\rangle$ problem.

In Hochster's work, the ring $\dfrac{R[x,y,z]}{(x^2 + y^2 + z^2 - 1)}$ is denoted by A where R denotes the real numbers, and he shows:

(i) $(A[P,Q])[t] \simeq S(E)[t]$ where P,Q are indeterminates and $S(E)$ is a suitable symmetric algebra.

(ii) $A[P,Q] \not\simeq S(E)$.

He also remarks that the only invertible elements in A are the real numbers.

Let us assume that there exists an isomorphism $\sigma: A[P,Q]\langle x\rangle \to S(E)\langle x\rangle$. Then the fact that A, hence $A[P,Q]$ and $S(E)$, are integral domains yields (by Corollary 3.2) that $\sigma(x) = ux^r$ where u is a unit in $S(E)$. The only invertible elements in $A[P,Q]$ are real numbers. If s is a real number $\neq 0,-1$ in $A[P,Q]$, then $\sigma(s)$ and $\sigma(1 + s) = 1 + \sigma(s)$ are both units, and we obtain by Corollary 3.2 that $\sigma(s)$ is in $S(E)$. Since $\sigma(x)$ and $\{\sigma(s) \mid s \in R\}$ generate the units of $S(E)\langle x\rangle$, we must have $r = \pm 1$ (where $\sigma(x) = ux^r$). In either case, we conclude as before that $A[P,Q] \simeq S(E)$, and this is a contradiction.

Hence $A[P,Q]\langle x\rangle \not\simeq S(E)\langle x\rangle$. In particular this tells us that $R[x] \simeq S[x]$ does not imply $R\langle x\rangle \simeq S\langle x\rangle$ in general.

CHAPTER 5

Some Isomorphism Problems

In this chapter, we consider several results related to the usual isomorphism problem for group rings, that is, when does $RG \simeq RH$ imply $G \simeq H$.

First of all, we consider the problem of "locating" subgroups H of G in the group ring RG. Sehgal [38] showed that if G is finite and $\alpha = \sum \alpha_g g \in ZG$ satisfies α central, $\alpha_1 = 1$, $\alpha^2 = m\alpha$ for some positive integer m and $\sum \alpha_g \neq 0$, then $\alpha = \sum_{g \in H} g$ for some $H \triangleleft G$. We extend this result to a class of integral domains of characteristic zero large enough to contain $\hat{Z}_p$ when G is a p-group. Then we indicate that in a very special case non-normal Sylow subgroups can be equationally defined in the above sense.

Secondly, we generalize a result of Sehgal in [37] from p-groups with property C to arbitrary finite p-groups. In the course of doing this, we obtain integers ℓ_i where ℓ_i is the number of conjugacy classes of elements x in G such that

(i) there exists $y \in G$ with $y^{p^i} = x$

(ii) no conjugate $z \neq y$ of y satisfies $z^{p^i} = x$.

We show that the ℓ_i are isomorphism invariants, that is, $Z_pG \simeq Z_pH$ implies the ℓ_i for G are equal to the ℓ_i for H. We also mention in this context an incomplete result of Raggi-Cardenas ([30],[31],[32]) concerning units in $Z_{p^n}G$ where G is a finite abelian p-group, and complete his result in the very special case $p = 2$.

Finally, we mention that Dade [11] has found two non-isomorphic finite metabelian groups G and H such that $\hat{Z}_pG \simeq \hat{Z}_pH$ for all primes p.

1. Normal Subgroups

The following result is due to Sehgal [38].

Theorem 5.1. Let G be a finite group and let $\alpha = \sum \alpha_g g$ be a central element of ZG such that $\alpha_1 = 1$, $\alpha^2 = m\alpha$ where m is a positive integer $\neq 0$, and $\sum \alpha_g \neq 0$. Then $\alpha = \sum_{g \in H} g$ for some $H \triangleleft G$.

We observe that this result can be extended to more general integral domains:

Proposition 5.2. Let R be a commutative integral domain of characteristic zero with 1. Let G be finite and let x be central in RG satisfying $x^2 = mx$ where m is an integer. Then the coefficients of x belong to $\frac{Z(\sqrt[e]{1})}{|G|}$ where e is the exponent of G.

Proof. Let $x = \sum \alpha_g g$ and choose g so $\alpha_g \neq 0$. Then $(xg^{-1})^{e+1} = m^e xg^{-1}$ since x is central. Viewing these elements in the regular representation of RG, the eigenvalues of xg^{-1} must consist of 0 and $m\eta_i$ where η_i are e'th roots of 1. However, we know that the trace of xg^{-1} is just $|G|\alpha_g$. Hence $|G|\alpha_g = m\sum \eta_i$ where the η_i are certain e'th roots of 1. Therefore, $\alpha_g = \frac{m\sum \eta_i}{|G|}$ and the proof is complete.

Note. Whenever we know that $\frac{Z(\sqrt[e]{1})}{|G|} \cap R = Z$, we can apply Theorem 5.1 to conclude that such an x (assuming also that $\sum \alpha_g \neq 0$ and $\alpha_1 = 1$) is necessarily of the form $\sum_{g \in H} g$ for some $H \lhd G$. For example, take G to be a finite p-group and $R = \hat{Z}_p$. Let $x \in \frac{Z(\sqrt[e]{1})}{|G|} \cap \hat{Z}_p$. Then $|G|x \in Z(\sqrt[e]{1}) \cap \hat{Z}_p$. Say $|G|x = \sum a_i \eta_i$ where $a_i \in Z$ and the η_i are e'th roots of 1.

Let $e = p^m$ and let ξ_i be a primitive p^i'th root of unity for $1 \leq i \leq m$. Consider the tower of fields $\hat{Q}_p \subset \hat{Q}_p(\xi_1) \subset \ldots \subset \hat{Q}_p(\xi_m)$ where $\hat{Q}_p$ is the field of p-adic numbers. Since the p^m'th cyclotomic polynomial is irreducible over $\hat{Q}_p$, we know that the irreducible polynomial of ξ_i over $\hat{Q}_p(\xi_{i-1})$ is $x^p - \xi_{i-1}$ for $i > 1$ and $x^{p-1} + x^{p-2} + \ldots + 1$ for $i = 1$.

Now $|G|x = \sum a_i \eta_i$. Taking the trace from $\hat{Q}_p(\xi_m)$ to $\hat{Q}_p(\xi_{m-1})$ yields $p|G|x = p \sum a_i \eta_i$ summed over all the η_i lying in $\hat{Q}_p(\xi_{m-1})$. This procedure continues nicely

until the last step, where the trace argument yields the result that $(p - 1)|G|x$ is an integer. Hence $|G|x$ is a rational number and an algebraic integer, therefore a rational integer. Since $|G|$ is a power of p and $x \in \hat{Z}_p$, we conclude that x is an integer. Hence, as a corollary of proposition 5.2 we have the following:

<u>Corollary 5.3</u>. Let G be a finite p-group and let $x = \sum \alpha_g g \in \hat{Z}_p G$ be central with $\alpha_1 = 1$, $\alpha^2 = m\alpha$ for some positive integer m, and $\sum \alpha_g \neq 0$. Then $\alpha = \sum_{g \in H} g$ for some $H \triangleleft G$.

<u>Notes</u>. 1. The tower of fields argument has been seen before in [41].

2. Since central idempotents have finite support, the conditions in the above three results identify the finite normal subgroups of an infinite group.

It is natural to ask whether it might sometimes be possible to identify non-normal subgroups in a similar manner. We show now that this can be done for groups of order $p^r q^s$ with one of the Sylow subgroups normal.

Let $c(\sum a_g g) = \sum a_g$. From now on, let $R = Z$.

<u>Proposition 5.4</u>. Let G be finite, $|G| = p^r q^s$, and say that $\sum_{g \in G} g = \alpha\beta$ where α is central, satisfies $\alpha^2 = p^r \alpha$, has coefficient of identity 1, and $c(\alpha) \neq 0$, and β satisfies $\beta^2 = q^s \beta$, has coefficient of identity 1

and $\beta = \beta^*$ (where $(\sum a_g g)^* = \sum a_g g^{-1}$). Then $\alpha = \sum_{g\in P} g$ where $P \triangleleft G$ is the p-Sylow subgroup of G and $\beta = \sum_{g\in Q} g$ where Q is a q-Sylow subgroup of G.

<u>Note</u>. If $P \triangleleft G$ is p-Sylow subgroup and Q a q-Sylow subgroup then $\sum_{g\in P} g$ and $\sum_{g\in Q} g$ certainly satisfy the conditions of Proposition 5.4.

<u>Proof</u>. We know from Theorem 5.1 that $\alpha = \sum_{g\in P} g$ where $P \triangleleft G$ is p-Sylow in G. It remains to prove the statement about β.

Let Q be a q-Sylow subgroup of G. Let $\hat{G}$ denote $\sum_{g\in G} g$, $\hat{Q}$ denote $\sum_{g\in Q} g$ and $\hat{P}$ denote $\sum_{g\in P} g$. Then $\hat{G} = \hat{P}\hat{Q} = \hat{Q}\hat{P}$. Also $\hat{G} = \hat{P}\beta$. Hence $\hat{P}(\hat{Q} - \beta) = 0$ and $\hat{Q} - \beta \in \Delta_Z(G,P)$.

Therefore, $\beta = \hat{Q} + m$ where $m \in \Delta_Z(G,P)$. Note that $\beta^2 = q^s\beta$ implies $(\hat{Q} + m)^2 = q^s(\hat{Q} + m)$ which, together with $\hat{Q}^2 = q^s\hat{Q}$, implies $q^s m = \hat{Q}m + m\hat{Q} + m^2$.

Also $\beta = \beta^*$ and $\hat{Q} = (\hat{Q})^*$ implies $m = m^*$.

Now $m \in \Delta_Z(G,P)$ implies that $m = \sum a_i(p_i - 1)$ with $a_i \in ZG$ and $p_i \in P$. Since each element in G can be written uniquely as $q_i p_j$ with $q_i \in Q$, $p_i \in P$, we can write $m = \sum b_{i,j}q_i(p_j - 1)$ with $b_{i,j} \in Z$, $q_i \in Q$ and $p_j \in P$. Therefore, $m = \sum b_{i,j}q_i p_j - \sum_i(\sum_j b_{i,j})q_i$.

Also $m^* = \sum b_{i,j}p_j^{-1}q_i^{-1} - \sum_i(\sum_j b_{i,j})q_i^{-1}$.

Therefore $m^2 = mm^*$ has coefficient of 1 equal to $\sum_{i,j} b_{i,j}^2 + \sum_i (\sum_j b_{i,j})^2$.

$\hat{Q}m$ has coefficient of 1 equal to $-\sum_{i,j} b_{i,j}$ and similarly for $m\hat{Q}$. $q^s m$ has coefficient of 1 equal to zero since $m = \beta - \hat{Q}$ and both β and $\hat{Q}$ have coefficients of identity equal to 1.

By recalling that $q^s m = \hat{Q}m + m\hat{Q} + m^2$, and using the above expressions for the coefficients of 1, we obtain $2\sum_{i,j} b_{i,j} = \sum_{i,j} b_{i,j}^2 + \sum_i(\sum_j b_{i,j})^2$. Since $\sum_i(\sum_j b_{i,j})^2 \geq \sum_i(\sum_j b_{i,j}) = \sum_{i,j} b_{i,j}$, we obtain $\sum_{i,j} b_{i,j} \geq \sum_{i,j} b_{i,j}^2$.

This implies that $b_{i,j} = 0$ or 1 for all i,j and, for each i, only one $b_{i,j} \neq 0$ (since $2\sum_{i,j} b_{i,j} = \sum_{i,j} b_{i,j}^2 + \sum_i(\sum_j b_{i,j})^2$).

Hence $m = \sum q_i(p_i - 1)$ for suitable $q_i \in Q$, $p_i \in P$ with all q_i different. Therefore, we get that $\beta = \sum g$ for suitable group elements $g \in G$. $\beta^2 = q^s\beta$ implies that the support of β forms a multiplicatively closed set and, since finite, a subgroup of G. The result follows.

2. Isomorphism Invariants of p-Groups

Definition. Let G be a finite p-group. Then ℓ_i is the number of (conjugacy) class sums K such that there exists a (conjugacy) class sum L with $L^{p^i} = K$ in Z_pG.

Equivalently, ℓ_i is the number of conjugacy classes K such that (i) if $x \in K$, there exists $y \in G$ such that $y^{p^i} = x$ and (ii) no conjugate z of y satisfies $z^{p^i} = x$.

Examples. $\ell_o = \alpha =$ the number of conjugacy classes in G. $\ell_n = 0$ when $n > e$, the exponent of G.

We will prove in this section that if G,H are finite p-groups with $Z_pG \simeq Z_pH$, then the numbers ℓ_i are the same for both G and H. We will first require a preliminary lemma and then a theorem describing the central units of Z_pG.

Lemma 5.5 [37]. Let K be a conjugacy class of a finite p-group G and let b be fixed. Then for $t > 0$, the number of solutions $(x_1,x_2,\ldots x_pt)$ of $x_1x_2\ldots x_pt = b$, $x_i \in K$, not all x_i equal, is a multiple of p.

Theorem 5.6. Let G be a finite p-group. Let $\Gamma_i(G) = \{y \in \mathfrak{z}(Z_pG) \mid y^{p^i} = 1\}$ for $i > 0$ where $\mathfrak{z}(Z_pG)$ is the centre of Z_pG. Then $|\Gamma_i(G)| = p^{\alpha - (\mathfrak{z}(G) : G_i) - \ell_i}$ where α is the number of conjugacy classes in G and $g_i = \{g \in \mathfrak{z}(G) \mid g^{p^i} = 1\}$.

Proof. As in [37], let $\mathfrak{Z}(G) = \bigcup_{s=1}^{m} \omega_s G_i$ with $\omega_1 = 1$ be a coset decomposition of $\mathfrak{Z}(G)$ with respect to G_i and let $G_i = \{g_1, g_2, \ldots g_n\}$. If $y \in \mathfrak{Z}(Z_pG)$, $y = \sum_{s,j} \alpha_{s,j}\omega_s g_j + \sum \beta_i K_i$, then $y^{p^i} = (\sum_{s,j} \alpha_{s,j}\omega_s g_j)^{p^i} + \beta_1^{p^i} K_1^{p^i} + \ldots + \beta_r^{p^i} K_r^{p^i}$ by Lemma 5.5, where $\alpha_{s,j} \in Z_p$ and $K_1, \ldots K_r$ are conjugacy class sums with more than one element. Note that each term $K_j^{p^i}$ is either zero or a conjugacy class sum itself by Lemma 5.5 and, hence, $K_j^{p^i} \neq 0$ if, and only if, $K_j^{p^i}$ satisfies the rules defining the numbers ℓ_i.

Hence, $y^{p^i} = 1$ if, and only if, $(\sum \alpha_{s,j} w_s g_j)^{p^i} = 1$ and, for a given conjugacy class L, $\sum_j \beta_j^{p^i} = 0$ summed over all j such that the corresponding $K_j^{p^i}$ are equal to L. The first condition amounts to $\sum_j \alpha_{s,j}^{p^i} = 1$ if $s = 1$ and 0 otherwise. This represents a loss of $(\mathfrak{Z}(G) : G_i)$ degrees of freedom. The second condition amounts to an additional loss of ℓ_i degrees of freedom. Hence $|\Gamma_i(G)| = p^{\alpha - (\mathfrak{Z}(G):G_i) - \ell_i}$.

Corollary 5.7. The ℓ_i are isomorphism invariants.

Proof. If $Z_pG \simeq Z_pH$, then $|\Gamma_i(G)| = |\Gamma_i(H)|$, $\mathfrak{Z}(G) \simeq \mathfrak{Z}(H)$ [37] and α is the same in both G and H (since the conjugacy class sums form a basis for the centre). Hence the ℓ_i are invariant.

In the case where G is a finite abelian p-group, the units of Z_pG have also been determined by Raggi-Cardenas ([30],[31],[32]). More generally, he claims to have found the structure of the unit group of $Z_{p^n}G$ for G a finite abelian p-group, but the proof of his main result seems to be incomplete. The breakdown occurs in the discussion of the following proposition:

Proposition 5.8. Let G be a finite abelian p-group. If $x = \sum \alpha_g g$ is a unit of order p in $Z_{p^2}G$, then every α_g except one is divisible by p.

It would be interesting to know whether the above proposition is true or false. We will show it to be true in the case $p = 2$.

Proposition 5.9. Let G be a finite abelian 2-group. If $x = \sum \alpha_g g$ is a unit of order 2 in Z_4G, then every α_g except one is divisible by 2.

Proof. We proceed by induction on $|G|$. If $|G| = 2$ the result is trivial, so we assume $|G| > 2$. Since $(a + 2b)^2 \equiv a^2(4)$, we may assume that all α_g are either 0 or 1. We may also assume that $\alpha_1 = 1$ and, in that case, since $(\sum h)^2 \equiv \sum h^2(2)$, there must be an odd number of group elements h of order ≤ 2 with $\alpha_h = 1$.

Choose $h_0 \in G$ of order 2 such that $\alpha_{h_0} = 0$ and let $H = \{1,h_0\}$. Such an h_0 exists by the above

argument. Since $|G/H| < |G|$ and $\bar{x}$ is a unit of order ≤ 2 in G/H, we have $\bar{x} = 1 + 2s$ for some s. We conclude that if $\alpha_g = 1$ and $g \neq 1$, then $\alpha_{gh_o} = 1$ also.

Now let g_o be of largest possible order such that $\alpha_{g_o} = 1$ and assume that $|g_o| > 1$. The term $2g_o$ appears in $(\sum \alpha_g g)^2$, and it must cancel with something. Hence we must have $g_o = \ell k$ with $\ell, k \neq 1, g_o$, $\ell \neq k$, and $\alpha_\ell = \alpha_k = 1$. By the argument in the last paragraph, $\alpha_{\ell h_o} = \alpha_{kh_o} = 1$ and $(\ell h_o)(kh_o) = \ell k = g_o$ also. Since both the pair (ℓ, k) and the pair $(\ell h_o, kh_o)$ contribute $2g_o$ to x^2, the original $2g_o$ term does not cancel out. This contradicts $x^2 = 1$. Hence $\{g | \alpha_g = 1\} = \{1\}$ and the proof is complete.

3. Isomorphism over p-adics

Whitcomb ([39],[43]) proved that if G, H are finite metabelian, then $ZG \simeq ZH$ implies $G \simeq H$. Dade [11] constructed two nonisomorphic finite metabelian groups G, H with $FG \simeq FH$ for all fields F. We would like to remark here that for Dade's groups, $\hat{Z}_pG \simeq \hat{Z}_pH$ for all primes p also. Hence, Dade's example yields two nonisomorphic finite metabelian groups with isomorphic p-adic integral group rings for all primes p. In particular, $\hat{Z}_pG \simeq \hat{Z}_pH$ for all $p \not\Rightarrow ZG \simeq ZH$ necessarily.

CHAPTER 6

Nonarchimedian Group Algebras

In this chapter, we consider the possibility of allowing infinite sums in a group algebra when the ring of coefficients has a valuation. For instance, $\ell_1(G) = \{\sum \alpha_g g \mid \alpha_g \in C, \sum|\alpha_g| < \infty\}$ where C is the field of complex numbers and $|\,|$ is the ordinary absolute value, is well known. We will study the rings $\ell(R,G) = \{\sum \alpha_g g \mid \alpha_g \in R, \lim \alpha_g = 0\}$ where R is $\hat{Z_p}$ or $\hat{Q_p}$ equipped with the p-adic valuation. Our primary interest is in the algebraic properties of these rings, and many of the results in this chapter can be obtained for $\ell(R,G)$ over more general valuation rings. In particular, if R is a ring and $I \underset{2}{\triangleleft} R$, we could form $\hat{R} = \varprojlim R/I^n$ and study $\ell(\hat{R},G)$.

Many of the arguments used for group rings can also be employed to advantage here. In such cases, we will only sketch the proofs involved.

1. Definitions and basic facts

Let G be a group, $\hat{Q_p}$ the field of p-adic numbers and $\hat{Z_p}$ the ring of p-adic integers. We define $\ell(\hat{Q_p},G) = \{\sum \alpha_g g \mid \alpha_g \in \hat{Q_p}, g \in G, \lim \alpha_g = 0\}$. By

$\lim \alpha_g = 0$, we mean that given any $\varepsilon > 0$, there exists a finite subset $\sigma(\varepsilon) \subseteq G$ such that $g \notin \sigma(\varepsilon)$ implies $v_p(\alpha_g) < \varepsilon$ where v_p is the p-adic valuation on $\hat{Q}_p$ written multiplicatively. It is straightforward to verify that $\ell(\hat{Q}_p,G)$ is a ring with pointwise equality, addition defined by $\sum \alpha_g g + \sum \beta_g g = \sum (\alpha_g + \beta_g)g$ and multiplication given by $(\sum \alpha_g g)(\sum \beta_g g) = \sum (\sum_{gh=k} \alpha_g \beta_h)k$. Define $\ell(\hat{Z}_p,G)$ to be the subring of $\ell(\hat{Q}_p,G)$ obtained by restricting the coefficients to be p-adic integers.

In general, let I be a 2-sided ideal of a ring R with 1. Let $\hat{R} = \varprojlim R/I^n$, that is, $\hat{R} = \{(\ldots,x_i, \ldots x_2,x_1) \in \prod_{n=1}^{\infty} R/I^n | x_i \in R/I^i, \alpha_i(x_i) = x_{i-1}$ for $i \geq 2\}$ where $\alpha_i : R/I^i \to R/I^{i-1}$ is the natural homomorphism. Define $\ell(\hat{R},G) = \{\sum \alpha_g g | g \in G, \alpha_g = (\ldots\alpha_{g,i},\ldots\alpha_{g,2},\alpha_{g,1})$ in $\hat{R}$, only finitely many $\alpha_{g,i} \neq 0$ for each $i\}$.

Most of our work will be done for the rings $\ell(\hat{Q}_p,G)$ and $\ell(\hat{Z}_p,G)$, but many of the results obtained in the p-adic integer case follow in the same way for $\ell(\hat{R},G)$.

The usual augmentation map $\varepsilon : \ell(\hat{Q}_p,G) \to \hat{Q}_p$ defined by $\varepsilon(\sum \alpha_g g) = \sum \alpha_g$ makes sense since $\sum \alpha_g$ converges (note that only countably many $\alpha_g \neq 0$ by the definition). In particular, if we write the α_g in their p-adic decomposition $\alpha_g = \sum_{T_g}^{\infty} \alpha_{g,i}p^i$, then $\sum \alpha_g$ is the limit of the partial sums $\sum_g \sum_{i=T_g}^{n} \alpha_{g,i}p^i$. The kernel of

the augmentation map, $\{\sum \alpha_g g \mid \sum \alpha_g = 0\}$, is called the augmentation ideal and is denoted by $\Delta(\ell(\hat{Q}_p,G))$. It is easy to see that $\Delta(\ell(\hat{Q}_p,G)) = \{\sum \alpha_g(g - 1) \mid \alpha_g \in \hat{Q}_p, \lim \alpha_g = 0\}$. Unlike group rings, however, Δ is not in general the ideal generated by $\{g - 1 \mid g \in G\}$. We will denote this latter ideal by $I(\ell(\hat{Q}_p,G))$. If H is a subgroup of G, we will require the right ideal generated by $\{h - 1 \mid h \in H\}$ and this will be denoted by $I_{rt}(\ell(\hat{Q}_p, G),H)$. If H is a normal subgroup of G, $I_{rt}(\ell(\hat{Q}_p,G),H)$ is a 2-sided ideal which will be denoted by $I(\ell(\hat{Q}_p,G),H)$. This is again in general different from $\Delta(\ell(\hat{Q}_p,G),H)$, the kernel of the natural epimorphism from $\ell(\hat{Q}_p,G)$ to $\ell(\hat{Q}_p,G/H)$.

<u>Theorem 6.1</u>. $\ell(\hat{Z}_p,G) \simeq \varprojlim Z_{p^n}G$.

<u>Proof</u>. Let $x = \sum \alpha_g g \in \ell(\hat{Z}_p,G)$ and $\alpha_g = \sum_{i=0}^{\infty} \alpha_{g,i}p^i \in \hat{Z}_p$. Define $\theta : \ell(\hat{Z}_p,G) \to \varprojlim Z_{p^n}(G)$ by $\theta(\sum \alpha_g g) = (\ldots, \sum (\sum_{j=0}^{s} \alpha_{g,j}p^j)g, \ldots \sum(\alpha_{g,1}p + \alpha_{g,0})g, \sum \alpha_{g,0}g)$. Since only finitely many α_g have $v_p(\alpha_g) \geq \varepsilon$ for any ε, the right hand side of the above equation lies in $\varprojlim Z_{p^n}(G)$. It is easily seen that θ is a ring isomorphism.

2. Chain Conditions

Proposition 6.2. If $\ell(\hat{Z}_p,G)$ is right Noetherian, then G has the maximal condition on subgroups. If $\ell(\hat{Q}_p,G)$ is right Noetherian, then G has the maximal condition on subgroups.

Proof. As in the group ring case, an increasing chain of subgroups H_i leads to an increasing chain of right ideals $I_{rt}(\ell(R,G),H_i)$ where $R = \hat{Z}_p$ or $\hat{Q}_p$.

Lemma 6.3 [14]. Let R be a commutative Noetherian ring and $I \underset{2}{\triangleleft} R$. Then $\varprojlim R/I^n$ is Noetherian.

Hence, we obtain:

Theorem 6.4. Let G be abelian. Then the following are equivalent:

(i) $\ell(\hat{Z}_p,G)$ is Noetherian

(ii) $\ell(\hat{Q}_p,G)$ is Noetherian

(iii) G is finitely generated.

Proof. Assume (i). Let $J_1 \subset J_2 \subset \ldots$ be a strictly ascending chain of ideals in $\ell(\hat{Q}_p,G)$. Then $J_1 \cap \ell(\hat{Z}_p,G) \subset J_2 \cap \ell(\hat{Z}_p,G) \subset \ldots$ is a strictly ascending chain of ideals in $\ell(\hat{Z}_p,G)$ since $x \in J_i - J_{i-1}$ implies $p^s x \in J_i \cap (\ell(\hat{Z}_p,G)) - J_{i-1} \cap \ell(\hat{Z}_p,G)$ for suitable s. This is not possible, so (ii) holds.

(ii) implies (iii) by Proposition 6.2.

Now assume (iii). Then $Z_{p^n}G$ is Noetherian for each n [10]. It follows from Lemma 6.3 and Theorem 6.1 that $\ell(\hat{Z}_p,G) \simeq \varprojlim Z_{p^n}G$ is Noetherian. This completes the proof of the theorem.

We proceed to investigate when $\ell(\hat{Q}_p,G)$ can be Artinian.

<u>Proposition 6.5</u>. The following conditions are equivalent:

(i) $\ell(\hat{Z}_p,G)$ is prime

(ii) $\ell(\hat{Q}_p,G)$ is prime

(iii) G has no finite normal subgroups.

<u>Proof</u>. Assume (i) is true. Let I,J be ideals of $\ell(\hat{Q}_p,G)$ with $IJ = 0$. Then $(I \cap \ell(\hat{Z}_p,G))(J \cap \ell(\hat{Z}_p,G)) = 0$. Hence either $I \cap \ell(\hat{Z}_p,G) = 0$ or $J \cap \ell(\hat{Z}_p,G) = 0$. It follows that $I = 0$ or $J = 0$.

Next assume (ii) to be true. If $H \triangleleft G$ and H is finite, then $I(\ell(\hat{Q}_p,G),H)(\sum_{h\in H} h) = 0$ and hence $I(\ell(\hat{Q}_p,G),H)<\sum h> = 0$. This is a contradiction.

Finally, assume (iii) to be true and let I,J be ideals of $\ell(\hat{Z}_p,G)$ with $IJ = 0$. In the homomorphic image $Z_pG \simeq \frac{\ell(\hat{Z}_p,G)}{(p)}$, we have $\bar{I}\,\bar{J} = 0$. Since Z_pG is prime

[10], $\bar{I} = 0$ or $\bar{J} = 0$. Hence $I \subseteq (p)$ or $J \subseteq (p)$.

Say $I \subseteq (p^r) - (p^{r+1})$ and $J \subseteq (p^s) - (p^{s+1})$ and let $I' = \{x \in \ell(\hat{Z}_p, G) \mid p^r x \in I\}$ and $J' = \{x \in \ell(\hat{Z}_p, G) \mid p^s x \in J\}$. Then I' and J' are ideals of $\ell(\hat{Z}_p, G)$, $I'J' = 0$ and $I' \not\subseteq (p)$, $J' \not\subseteq (p)$. This is a contradiction unless $I = 0$ or $J = 0$.

Using Proposition 6.5 and copying the proof of the group ring case [10], we obtain the result for Artinian rings:

<u>Theorem 6.6</u>. $\ell(\hat{Q}_p, G)$ is Artinian if, and only if, G is finite.

3. <u>Radicals</u>

In this section, we mention the prime radical and the Jacobson radical of $\ell(\hat{Z}_p, G)$ and $\ell(\hat{Q}_p, G)$. The study of the Jacobson radical of $\ell(\hat{Q}_p, G)$ seems to be difficult, but the problem of determining the Jacobson radical of $\ell(\hat{Z}_p, G)$ reduces to a group ring problem by the following:

<u>Proposition 6.7</u>. Let $\{R_i\}_{i=1}^{\infty}$ be rings with 1 and $\alpha_i : R_i \to R_{i-1}$ be epimorphisms. Then $J(\varprojlim R_i) = \varprojlim J(R_i)$ where by $\varprojlim J(R_i)$ we mean $\{(\ldots r_n, \ldots r_2, r_1) \in \varprojlim R_i \mid r_m \in J(R_m)$ for $1 \le m < \infty\}$.

<u>Proof</u>. Since each R_i is a homomorphic image of $\varprojlim R_i$, $J(\varprojlim R_i) \subseteq \varprojlim J(R_i)$ follows with no difficulty. Conversely, observe that $\varprojlim J(R_i)$ is an ideal of

$\varprojlim R_i$, namely the ideal consisting of all sequences $(\ldots x_n,\ldots x_2,x_1)$ with $x_i \in J(R_i)$ for each i. Let y_i be the right-quasi-inverse of x_i for each i, $1 \leq i < \infty$. Then we see that $(\ldots x_n,\ldots x_2,x_1) + (\ldots y_n,\ldots y_2,y_1) + (\ldots x_n,\ldots x_2,x_1)(\ldots y_n,\ldots y_2,y_1) = 0$ and we observe also that if $x_n + y_n + x_n y_n = 0$, then $\alpha_n(x_n + y_n + x_n y_n) = 0$, hence $\alpha_n(x_n) + \alpha_n(y_n) + \alpha_n(x_n)\alpha_n(y_n) = 0$ where α_n is the given epimorphism. It follows that $x_{n-1} + \alpha_n(y_n) + x_{n-1}\alpha_n(y_n) = 0$ and, therefore, that $y_{n-1} = \alpha_n(y_n)$ by the uniqueness of the right-quasi-inverse. Hence $(\ldots y_n,\ldots y_2,y_1) \in \varprojlim R_i$ and we conclude that $\varprojlim J(R_i)$ is a right-quasi-regular ideal and hence is contained in $J(\varprojlim R_i)$. This completes the proof.

Note. The containment $\subseteq$ holds for any radical property T, that is, $T(\varprojlim R_i) \subseteq \varprojlim T(R_i)$. The converse is not always true. For example, $0 = \beta(\hat{Z_p}) \neq (p) = \varprojlim \beta(Z_{p^n})$.

Corollary 6.8. $J(\ell(\hat{Z_p},G)) = \varprojlim J(Z_{p^n}G)$.

Proof. Theorem 6.1.

The prime radical case seems even less clear for $\ell(\hat{Z_p},G)$ and $\ell(\hat{Q_p},G)$. It is well known that $\ell_1(G)$ is semiprime and it follows from [10] that $\hat{Z_p}G$ and $\hat{Q_p}G$ are semiprime, but it is not known whether or not $\ell(\hat{Z_p},G)$ (equivalently $\ell(\hat{Q_p},G)$) is semiprime. The usual group

ring reduction goes through, so we may assume that G is a torsion FC-group with an element of order p.

4. Idempotents

Proposition 6.9. Let G be locally nilpotent. Then $\ell(\hat{Z}_p,G)$ has a nontrivial idempotent if, and only if, G contains an element of order prime to p.

Proof. Let $e = (\ldots a_n, \ldots a_2, a_1)$ be a nontrivial idempotent in $\ell(\hat{Z}_p,G)$, where we have identified $\ell(\hat{Z}_p,G)$ with $\varprojlim Z_{p^n}G$ using Theorem 6.1. If $a_1 = 0$, then $e \in J(\ell(\hat{Z}_p,G))$ since $(p) \subseteq J(\ell(\hat{Z}_p,G))$ which is impossible. If $a_1 = 1$, then e is a unit in $\ell(\hat{Z}_p,G)$ which is impossible. Hence a_1 is a nontrivial idempotent in Z_pG. By a result of Bovdi and Mihovski [5], G has an element of order prime to p.

The converse is trivial since if $(o(g),p) = 1$, then $\dfrac{1 + g + \ldots + g^{o(g)-1}}{o(g)}$ is a non-trivial idempotent in $\ell(\hat{Z}_p,G)$.

One of the nicest results about $\ell_1(G)$ is a result of Rider [33] which states that every central idempotent in $\ell_1(G)$ has finite support group (If $x =$

$\sum \alpha_g g$, the support group of x is the subgroup of G generated by $\{g \mid \alpha_g \neq 0\}$).

For $\ell(\hat{Z}_p, G)$, we obtain the following:

Theorem 6.10. Every central idempotent in $\ell(\hat{Z}_p, G)$ has finite support group.

Proof. Let e be a central idempotent in $\ell(\hat{Z}_p, G)$ and assume $e \neq 0,1$. Then $\bar{e}$ is a central idempotent in $Z_p G \simeq \dfrac{\ell(\hat{Z}_p, G)}{(p)}$ and, by the argument used in Proposition 6.9, $\bar{e}$ is nontrivial. Hence $\bar{e}$ has finite support group H ([6]). Since $Z_{p^{n-1}}H \simeq \dfrac{Z_{p^n}H}{(p^{n-1})}$ and (p^{n-1}) is nilpotent in $Z_{p^n}H$ we may lift $\bar{e}$ to an idempotent $u = (\ldots u_n, \ldots u_2, u_1)$ in $\hat{Z}_p H = \ell(\hat{Z}_p, H) \simeq \varprojlim Z_{p^n}H$ by lifting it one step at a time.

Now we observe that $e = u$ since they are commuting idempotents with $e - u \in (p) \subseteq J(\ell(\hat{Z}_p, G))$. Hence e has finite support group H.

5. Units

We show in this section that the problem of determining the units of $\ell(\hat{Q}_p, G)$ reduces to a problem concerning the group rings $Z_{p^n}G$. First observe that if $x, y \in \ell(\hat{Q}_p, G)$ and $xy = 1$, then for some integers r, s, we have $p^r x \in \ell(\hat{Z}_p, G)$, $p^s y \in \ell(\hat{Z}_p, G)$ and $(p^r x)(p^s y) = p^{r+s}$.

Conversely, if $x, y \in \ell(\hat{Z}_p, G)$ satisfy $xy = p^r$ for some r, then $(p^{-r}x)y = 1$. Hence the problem of determining the units of $\ell(\hat{Q}_p, G)$ is equivalent to the problem of determining solutions to the equations $xy = p^r$ in $\ell(\hat{Z}_p, G)$.

Theorem 6.11. Let $x \in \ell(\hat{Z}_p, G)$. Then the following are equivalent:

(i) there exists $y \in \ell(\hat{Z}_p, G)$ such that $xy = p^r$

(ii) there exists $y \in Z_{p^{r+1}}G$ such that

$$\bar{x}y = p^r \quad \text{where} \quad \bar{x} \in \frac{\ell(\hat{Z}_p, G)}{(p^{r+1})} \simeq Z_{p^{r+1}}G.$$

Proof. Clearly, if $xy = p^r$ for some y, then $\bar{x}\,\bar{y} = p^r$ in $Z_{p^{r+1}}(G)$ so (i) $\Rightarrow$ (ii) is obvious.

Assume now that there exists $y \in \ell(\hat{Z}_p, G)$ with $\bar{x}\,\bar{y} = p^r$ in $Z_{p^{r+1}}G$, i.e. $xy = p^r + p^{r+1}s$ where $s \in \ell(\hat{Z}_p, G)$. Thus $xy = p^r(1 + ps)$. Since $(p) \subseteq J(\ell(\hat{Z}_p, G))$, $1 + ps$ is invertible and hence there exists z in $\ell(\hat{Z}_p, G)$ with $xyz = p^r$.

Corollary 6.12. x is a unit in $\ell(\hat{Z}_p, G)$ if, and only if, $\bar{x}$ is a unit in Z_pG.

6. Residual Nilpotence

In this section, we prove that if G is a finitely

generated torsion-free nilpotent group, then $\bigcap_n \Delta^n(\ell(\hat{Q}_p,G)) = 0$. This result for group rings over the rational number field goes back to Jennings [21]. We use the induction technique of Formanek [12] and present the proof for G an infinite cyclic group. The rest of the argument, being identical with that of Formanek, will be omitted.

Theorem 6.13. Let $G = \langle g \rangle$ be an infinite cyclic group. Then $\bigcap_n \Delta^n(\ell(\hat{Q}_p,G)) = 0$.

Proof. Suppose $x \in \bigcap_n \Delta^n(\ell(\hat{Q}_p,G))$. Using the identity $y^2 - 1 = 2(y - 1) + (y - 1)^2$, we can write

$$x = \sum_1^\infty \alpha_i(g - 1)^i + \sum_1^\infty \beta_i(g^{-1} - 1)^i = \sum_1^\infty \alpha_i(g - 1)^i + \sum_1^\infty \beta_i(-1)^i g^{-i}(g - 1)^i = \sum_1^\infty (\alpha_i + (-1)^i\beta_i g^{-i})(g - 1)^i.$$

Also, since $\Delta^n(\ell(\hat{Q}_p,G)) = (g - 1)^n \ell(\hat{Q}_p,G)$, we can write $x = (g - 1)^n t_n$ for each n.

Because $g - 1$ is not a zero divisor, we conclude that $\alpha_1 - \beta_1 g^{-1} \in \Delta(\ell(\hat{Q}_p,G))$ and hence $\alpha_1 - \beta_1 = 0$. Therefore, $\alpha_1 - \beta_1 g^{-1} = \beta_1 g^{-1}(g - 1)$.

Again, dividing by $(g - 1)^2$, we obtain that $\beta_1 g^{-1} + \alpha_2 + \beta_2 g^{-2} \in \Delta(\ell(\hat{Q}_p,G))$ and hence $\alpha_2 + \beta_2 = -\beta_1$. Continuing this procedure, we get at the next step $\alpha_3 - \beta_3 = \beta_1 + 2\beta_2$.

In general, we may conclude that $\alpha_n + (-1)^n\beta_n = r_{n,1}\beta_1 + r_{n,2}\beta_2 + \ldots + r_{n,n-1}\beta_{n-1}$ where the $r_{i,j}$ are integers satisfying

(i) $r_{n,1} = (-1)^{n-1}$

(ii) $r_{n,s} + r_{n-1,s} = -r_{n-1,s-1}$ for $2 \le s \le n - 2$

(iii) $r_{n,n-1} = (-1)^{n-1}(n - 1)$.

We proceed to prove that these relations imply that $\beta_i = 0$ for all i using the fact that only finitely many β_i can have p-adic value greater than $\frac{1}{p^s}$ for any fixed s.

Given any natural number s, we can choose $n = n_s$ such that $v_p(\alpha_m) \le \frac{1}{p^s}$ and $v_p(\beta_m) \le \frac{1}{p^s}$ for all $m \ge n$. Then $v_p(\alpha_{n+1} + (-1)^{n+1}\beta_{n+1} + \alpha_n + (-1)^n\beta_n) \le \frac{1}{p^s}$. Now, $\alpha_{n+1} + (-1)^{n+1}\beta_{n+1} + \alpha_n + (-1)^n\beta_n = r_{n+1,1}\beta_1 + r_{n+1,2}\beta_2 + \ldots + r_{n+1,n-1}\beta_{n-1} + r_{n+1,n}\beta_n + r_{n,1}\beta_1 + r_{n,2}\beta_2 + \ldots + r_{n,n-1}\beta_{n-1} = -r_{n,1}\beta_2 - r_{n,2}\beta_3 \ldots -r_{n,n-2}\beta_{n-1} + r_{n+1,n}\beta_n$ (by (i) and (ii)). Also, $\alpha_{n+2} + (-1)^{n+2}\beta_{n+2} + \alpha_{n+1} + (-1)^{n+1}\beta_{n+1} = -r_{n+1,1}\beta_2 - r_{n+1,2}\beta_3 \ldots -r_{n+1,n-2}\beta_{n-1} + \gamma$ where $v_p(\gamma) \le \frac{1}{p^s}$.

Adding, we conclude by using (i) and (ii) that $\alpha_n + (-1)^n\beta_n + 2(\alpha_{n+1} + (-1)^{n+1}\beta_{n+1}) + \alpha_{n+2} + (-1)^{n+2}\beta_{n+2} = r_{n,1}\beta_3 + r_{n,2}\beta_4 + \ldots + r_{n,n-3}\beta_{n-1} + \gamma$ where $v_p(\gamma) \le \frac{1}{p^s}$.

Proceeding in this manner, we obtain an expression for β_{n-1} as an integral linear combination of terms each having p-adic value $\leq \frac{1}{p^s}$. It follows that $v_p(\beta_{n-1}) \leq \frac{1}{p^s}$. Working back, we conclude that $v_p(\beta_i) \leq \frac{1}{p^s}$ for all i. Since s is arbitrary, we can conclude that $\beta_i = 0$ for all i. It is easy to see that all $\alpha_i = 0$. Therefore $x = 0$.

REFERENCES

1. S.S. Abhyankar, P. Eakin and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring (to appear).

2. S.A. Amitsur, Radicals of polynomial rings, Canad. J. Math 8 (1956), 355-361.

3. M. Auslander and R.C. Lyndon, Commutator subgroups of free groups, Amer. J. Math. 77 (1955), 929-931.

4. A.A. Bovdi, The intersection of the powers of the fundamental ideal of an integral group ring, Mat. Zametki 2 (1967), 129-132.

5. A.A. Bovdi and S.V. Mihovski, Idempotents in crossed products, Soviet Math. Dokl. 11 (1970), 6, 1439-1441.

6. R.G. Burns, Central idempotents in group rings, Can. Math. Bull. 13 (4) (1970), 527-528.

7. D.B. Coleman, Idempotents in group rings, Proc. Amer. Math. Soc. 17 (1966), 962.

8. D.B. Coleman, On group rings, Canad. J. Math. 22 (1970), 249-254.

9. D.B. Coleman and E.E. Enochs, Isomorphic polynomial rings, Proc. Amer. Math. Soc. 2 (1971), 247-252.

10. I.G. Connell, On the group ring, Canad. J. Math. 15 (1963), 650-685.

11. E.C. Dade, Deux groupes finis distincts ayant la même algèbre de groupe sur tout corps, Math. Z. 119 (1971), 345-348.

12. E. Formanek, A short proof of a theorem of Jennings, Proc. Amer. Math. Soc. 26 (1970), 405-407.

13. R.W. Gilmer Jr., R-automorphisms of R[x], Proc. London Math. Soc. (3) 18 (1968), 328-336.

14. S. Greco and P. Salmon, Topics in m-adic Topologies, Springer-Verlag 1971.

15. K.W. Gruenberg, The residual nilpotence of certain presentations of finite groups, Arch. Math. 13 (1962), 408-417.

16. K.W. Gruenberg and J.E. Roseblade, The augmentation terminals of certain locally finite groups, Canad. J. Math. (to appear).

17. P. Hall, Nilpotent Groups, Lecture notes Canadian Math Congress summer seminar 1957.

18. B. Hartley, The residual nilpotence of wreath products, Proc. London Math. Soc. (3) 21 (1970), 385-398.

19. M. Hochster, Nonuniqueness of coefficient rings in a polynomial ring (to appear).

20. D. Jacobson, The isomorphism problem for polynomial rings, Notices Amer. Math. Soc. 18 (5) (1971), 687-16-2.

21. S.A. Jennings, The group ring of a class of infinite nilpotent groups, Canad. J. Math. 7 (1955), 169-187.

22. I. Kaplansky, Infinite Abelian Groups, University of Michigan Press 1954.

23. W. Magnus, Beziehungen zwischen gruppen und idealen in einem speziellen Ring, Math. Ann. 111 (1935), 259-280.

24. A.I. Mal'cev, Generalized nilpotent algebras and associated groups, Mat. Sbornik N.S. 25 (67) (1949), 347-366. Amer. Math. Soc. Transl. (2) 69 (1968).

25. J.N. Mital, On residual nilpotence, J. London Math. Soc. 2 (1970), 337-345.

26. I.B.S. Passi, Polynomial maps on groups, Ph.D. thesis, University of Exeter, 1966.

27. I.B.S. Passi, Polynomial maps on groups, J. Algebra 9 (1968), 121-151.

28. I.B.S. Passi, Dimension subgroups, J. Algebra 9 (1968), 152-182.

29. D.S. Passman, Nil ideals in group rings, Mich. Math. J. (1962), 375-384.

30. F.F. Raggi Cardenas, Las unidaden en anillos de grupo I, An. Inst. Mat. Univ. Nac. Autónoma México 7 (1967), 27-35.

31. F.F. Raggi Cardenas, Las unidades en anillos de grupo II, An. Inst. Mat. Univ. Nac. Autónoma México 8 (1968), 91-103.

32. F.F. Raggi Cardenas, Las unidades en anillos de grupo con coefficientes en K_{p^n}, Z_{p^n} y $\hat{Z}_p$, An. Inst. Mat. Univ. Nac. Autónoma México 10 (1970), 29-66.

33. D. Rider, Central idempotents in group algebras, Bull. Amer. Math. Soc. 72 (1966), 1000-1002.

34. R. Sandling, The modular group rings of p-groups, Ph.D. Thesis, University of Chicago, 1969.

35. R. Sandling, Subgroups dual to dimension subgroups, Proc. Camb. Phil. Soc. 71 (1972), 33-38.

36. W.R. Scott, Group Theory, Prentice-Hall, 1964.

37. S.K. Sehgal, On the isomorphism of group algebras, Math. Z, 95 (1967), 71-75.

38. S.K. Sehgal, On the isomorphism of integral group rings I, Can. J. Math. 21 (1969), 410-413.

39. S.K. Sehgal, On the isomorphism of integral group rings II, Can. J. Math. 21 (1969), 1182-1188.

40. S.K. Sehgal, Units in commutative integral group rings, Math. J. Okayama Univ. 14 (1970), 135-138.

41. S.K. Sehgal, Isomorphism of p-adic group rings, J. Number Theory 2 (1970), 500-508.

42. P.F. Smith, On the intersection theorem, Proc. London Math. Soc. (3) 21 (1970), 385-398.

43. A. Whitcomb, The group ring problem, Ph.D. Thesis, University of Chicago, 1968.

44. H. Zassenhaus, Ein Verfahren, jeder endlichen p-Gruppe einen Lie-Ring mit der Charakteristik p zuordnen, Abh. Math. Sem. Hamburg Univ. 13 (1939), 200-207.